ON THE

# FREE MOTION OF POINTS,

AND ON

# UNIVERSAL GRAVITATION,

INCLUDING

THE PRINCIPAL PROPOSITIONS OF BOOKS I. AND III.

OF

## THE PRINCIPIA;

*THE FIRST PART*

OF

## A TREATISE ON DYNAMICS.

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*THIRD EDITION.*

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# PREFACE

## TO THE SECOND EDITION.

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By the publication of the former edition of the 'Treatise on Dynamics', I consider myself as having contracted a sort of obligation to present the subject to the Students of the University in the best form I can devise, so long as the work is in use. Otherwise I do not think that with my present occupations and engagements I should have ventured upon a task which now appears to me so difficult and so responsible as the composition of such a Treatise. A few years experience has a great tendency to diminish the confidence of producing what shall satisfy himself and others, with which a young author sets out: and he learns that the vivid impression of the fancied deficiencies and imperfections of preceding works which at first induced him to write, is a very insufficient warrant of his own skill and judgement.

I will say a few words as to the views with which I have prepared the present edition.

A leading object in compiling a Treatise on Dynamics for the use of this University must of course be to conduct the Student through most of the reasonings, formulæ and propositions which are requisite as a preparation for the higher investigations connected with physical science, and especially with the most profound and perfect of mathematical sciences, Physical Astronomy. From the beauty, the importance, and the celebrity of this portion of mathematics, it has necessarily been long the ultimate goal and aim of the best mathe-

maticians; and there can be no doubt that, so far as is practicable, our study of the science of Mechanics should from the first be regulated with the view of making it a fitting introduction to the mechanics of the universe. Till recently however it has not been easy to effect this. The admirable systematical treatises of Laplace and Lagrange, and still more the various memoirs which contain the original labours of those and other great mathematicians on this subject, are not suited to the common Student, and can never be familiarly consulted or fully mastered except by accomplished and persevering analysts. When the first edition of the present Treatise was published, the only English work on the subject which professed to be elementary was the "Physical Astronomy" of the late Professor Woodhouse. Every one acquainted with the recent history of Cambridge studies, will be ready to pay his tribute of respect to the memory of Mr Woodhouse, and to assign to him the high praise of having been the first to introduce mathematics among us in the form which the advancing researches of continental mathematicians had given to the science. He laboured long almost alone to promote this object; but the change to which he gave the first impulse has eventually taken place in the fullest manner. Like his other works, his Physical Astronomy contains the most ample evidence of great thought, great learning, and a strong wish to be practically useful. But I do not think that it has been found a convenient book for the Cambridge Student, and it appears to me difficult to introduce any considerable portion of it into the usual course of reading. Within the last few years however the deficiency which existed in this respect has been admirably supplied by Professor Airy's "Mathematical Tracts;" and those who wish to pursue these investigations still further, can now have recourse to Mrs Somerville's "Mechanism of the Heavens."

Without speaking of the other merits of Professor Airy's Tracts, I may observe that by their plan, and especially by their subdivision into Propositions, they are peculiarly adapted to our course of study. The student of this magnificent subject on a wider scale will find reason to acknowledge great obligations to Mrs Somerville for having put in a convenient and compendious form the results of some of the most modern improvements in its investigations. And most readers will, I think, allow that the unparalleled occurrence of our owing such a work to a female pen gives an additional pleasure to our gratitude. Nor are the circumstances which so much attract our admiration to this work without their bearing on its usefulness here. Our willingness to adopt a more extended study of the mechanism of the heavens into our academic system must needs increase, when these severer studies, thus shewn to be reconcilable with all the gentler train of feminine graces and accomplishments, can no longer, with any shew of reason, be represented as inconsistent with a polished taste and a familiar acquaintance with ancient and modern literature.

Having before me such books of instruction for the higher parts of the science, I have endeavoured to lead the Student up to them, and have given a few of the introductory steps of the Lunar and Planetary Theories, so as to place him at the point from which he may proceed under the auspices of these worthier guides, to whom I have in each of these cases finally referred him. In this part of the work, I have introduced several of the analytical investigations of Laplace and other writers on the subject; as the developement of $v$ and $r$ in terms of $t$; (Art. 32—35;) the curious theorems of Lambert concerning the ellipse and parabola, which are of use in the problem of the orbit of a comet; (Art. 36—38, and 41;) and Pontecoulant's elegant integration of the equa-

fraction which a month is of a year;) and this method might be made to give the annual equation of the motion of the node, (p. 146,) which determinations would require several steps to obtain them from the equations. Indeed it is not difficult to see the reason of this property of the geometrical method. In the symbolical process, the terms are arranged according to the powers of a small quantity, and no term can be rejected without ascertaining analytically that the rejection will not affect the powers which are retained: but in the geometrical method the extent to which an inequality will be affected by other inequalities is judged of by the nature of its dependence on the forces producing it, and not by the symbols by which it is denoted. It may easily happen that, by considering this dependence, we obtain an accuracy which it would require several terms to express.

The motion of the nodes, found by Newton's method, in Art. 84, and agreeing with the analytical result as far as the term involving $m^3$, is the value when we leave out of consideration the effect of the change of the radius vector, and the acceleration of areas produced by the disturbing forces. But Newton has also considered the effect of these disturbances of the second order in the motion of the nodes (Art. 83). And it has been shewn by Plana (*Zach*, Corresp. Astron. Vol. XVI,) that, when the corrections given by his method are properly applied, the results are correct as far as the term involving $m^4$. The variation of the radius vector of the Moon's orbit from $1+x$ to $1-x$ diminishes the mean motion of the nodes in the ratio 1 to $1-2x$ (Art. 83.): and the unequal description of areas produces a further diminution in the ratio 1 to $1-\frac{3m^2}{4}$. And as the mean motion of the nodes is $\frac{3m^2}{4}$ of the Moon's true motion, (neglecting ulterior terms,) the correction of

the mean motion arising from the causes just mentioned is $\frac{3m^2}{4}\left(2x+\frac{3m^2}{4}\right)$, as stated by Laplace and Plana*.

I have not introduced this correction in the text, though it is applied by Newton, (Book III. Prop. 31,) because it has not generally made part of the reading of the Principia in this place; and my object was not to introduce new employment of this kind; since, however curious such investigations might be, they must be considered as deviations in the path of the student who wishes to make a systematic advance in cosmical Dynamics. For the same reason I have omitted some other propositions of the Third Book.

It is the more necessary not to devote too much time to the geometrical method, since not only is that method soon stopt by insurmountable obstacles, but we cannot be sure of obtaining even a first approximation by means of it. Thus the greatest periodical inequality of the Moon's motion, the Evection, and the great permanent inequality, the progression of her apsides, cannot be calculated at all, even loosely, as to quantity, by the methods given by Newton, or by any of his followers.

But in the cases where Newton has explained his mode of finding the amount of the Lunar inequalities, I have thought I should be doing a service to the English reader in offering to him such propositions, retaining the reasoning of the Principia, but expressing it in such a manner as to facilitate a comparison of the results thus obtained with those obtained

* It may be mentioned also that Professor Woodhouse and Mr Lubbock have shewn that Newton's value of the horary motion of the node

$$\delta N = 3m^2 \delta\theta \cos.(\theta'-\theta) \sin.(\theta-N) \sin.(\theta'-N),$$

(see p. 144 of this volume,) results from the equations obtained in the analytical investigation of the subject and is true for all powers of the disturbing force. See Physical Astronomy, p. 441, 442. Phil. Trans. 1832. Part I. p. 39.

by the analytical methods, principally as given by Professor Airy. This comparison appears to me extremely curious. I have instituted it both to illustrate the more vague and general proof of the *existence* and form of the lunar inequalites, which Newton gives in the 66th proposition of the First Book; and the more precise calculation of their *quantities*, which occupies the greater part of the Third Book.

No one I think can study this portion of the Principia without considering it, as Laplace calls it, one of the most remarkable parts of that unrivalled work. For my own part, though I previously had always admired Newton as the greatest of philosophers, my admiration of him has received no slight accession from this examination of his immortal labours. Men have often a tendency to represent the greatest discoverers as in a considerable degree the products of their age;—as the persons who conceive in a distinct form that which was floating vaguely in the minds of their contemporaries; who first utter the word which was rising to the lips of many. Such opinions have sometimes been entertained of Newton; most inadequate notions, as appears to me, of his true place and character. If we allow that some half-anticipations of this kind were in existence with regard to the doctrine of the gravitation of revolving bodies to their central ones, let us observe how much remains to elevate him above the stature of common men;

ἔξοχος Ἀργείων κεφαλήν τε καὶ εὐρέας ὤμους.

To prove that, by a central force varying inversely as the square of the distance, bodies would describe ellipses according to Kepler's Laws, was a step of geometry which was necessary for the verification of the conjecture of *central* gravitation; and this step it does not appear that any other geometer of that time could make. But Newton's progress only began here.

The advance from this to *universal* gravitation was all his own; and the confirmation of this vast thought required the establishment of another set of curious and difficult geometrical propositions (Prop. 71, & seqq.) And when he had reached this lofty summit of truth, the manner in which he darted down again upon all the particular consequences which the general principle involved, has no parallel, so far as I have ever read, in the history of the mind of man, either with regard to the sagacity with which he perceived the nature of the effects, or the mathematical power with which he traced their details. All the most various and apparently incoherent phenomena were arranged at once in their true places. The different inequalities of the Moon's motion which had been gradually and painfully detected, one by one, century after century; the slow precession of the equinoxes among the stars; the ancient puzzle of the tides; the changed rate of a pendulum carried to the equator; these known facts, and other appearances, no less real, but which men were now first taught to see, were all conceived in his mind as manifestations of one single law; and their circumstances defined with a fertility and beauty of mathematical resource which have never been surpassed. If any one would compare Newton with other men, let him set aside for a moment his greatest glory, the discovery of universal gravitation, and supposing the law given, let him ask what one, or two, or three mathematicians of that time, or of any time, beginning from the point where he began and using his instruments, could have deduced from that law the train of consequences so remarkably verified in the material world, which are presented in the Third Book of the Principia.

It adds to the extraordinary character of this book, and it adds also to its difficulty, that it has the appearance of having been written with great rapidity. The propositions are very far from being presented in that concinnity of synthesis

with which a very little additional trouble might have invested them; and many of the calculations are thrown together with small regard to order, fulness or clearness. The author has not time to explain all his reasonings: thus he takes a step "ob causam quam hic exponere non vacat:" (Book III. Prop. 23): in one of the most complex of his investigations (Prop. 31) he gives the conclusion "uti rationem ineunti facile patebit:" and after calculating some of the lunar inequalities, and explaining his methods, he gives the results of the calculation of several others with little or no explanation, in the Scholium after Prop. 35. "Per eandem theoriam inveni præterea" is the way in which he states the amount of the annual equation. "Inveni etiam" is the introduction to the annual equation of the motion of the apogee and of the nodes: "per theoriam gravitatis constitit etiam" is the phrase for the "equatio semestris:" "uti ex theoriâ gravitatis colligo" for the "semestris secunda;" and the effect of the evection, with some other inequalities, is very briefly stated in the same scholium.

It has often been asserted, sometimes in sorrow and sometimes in anger, that the countrymen of Newton have done little or nothing for that sublime science of which he was the founder. The charge is not without its share of truth; and it can hardly be urged in defence that their faculties were absorbed in the study of their great teacher. For if they had employed themselves in verifying the above calculations and presenting the investigations in a distinct and accessible form, their labours would have been, at an earlier period, of considerable value; and the geometrical method of treating the problem of the three bodies might have had its triumphs to point to, as well as the analytical. I am not aware that this has been done in any degree, or that any English commentator on Newton has yet attempted to simplify and explain the Third Book of the Principia. We cannot

consider as an important exception to this remark, the mode of finding the mean motion of the Moon's nodes, assuming the horary motion as determined in the Principia, which Machin and Pemberton separately invented, and which is published by Newton himself in the third edition of the Principia. Yet with such exceptions, neither in the period immediately succeeding the publication of the Newtonian theory, nor in more recent times, have the calculations founded upon it been elucidated or extended by Englishmen. Before 1750, when Clairaut rightly explained the supposed discrepancy between the observed amount of the motion of the Moon's apsides and that resulting from theory, several English writers (Machin, Walmesley, Murdock) had pretended to deduce the true quantity from the Newtonian doctrine; but inasmuch as they neglected the transverse disturbing force, we know that there must have been some mathematical fallacy in their deduction. Thomas Simpson, who makes this observation, had obtained the solution of the question in a more analytical manner, but was anticipated in his publication of the result by Clairaut, who announced his own success in the same research. Since that time we can scarcely mention any English attempts to illustrate or carry forward this subject. The second volume of the Astronomy of the late Professor Vince contains several parts of the theory of the Moon, "in which" says he, "we shall principally follow the indirect methods of Newton and Frisi;" but I believe few mathematicians have been tempted to go through the labour of mastering and verifying the calculations in that work.

In the present volume I have of course not followed the train of propositions by which Newton investigates the quantity of the precession of the equinoxes. According to the division of the science here adopted this subject belongs to the Third Book: but perhaps the manner in which it is treated

in Professor Airy's Tracts may supersede the necessity of my entering into detail with respect to this problem. I have also omitted the investigations concerning the determination of the orbits of comets; a subject which from its extent and complexity might properly be made the business of a separate tract.

I have in most cases adopted Newton's numbers; for any additional accuracy of calculation will probably be now rather sought by other methods; and if any one wishes to compare my results with the text of the Principia, this agreement will facilitate the reference. In cases however where Newton's numbers are widely wrong, as in the instance of the mass of the Moon, I have substituted the more correct modern determinations.

Besides the Third Book of the Principia, I have introduced all the propositions in the First Book, which can, according to my judgment, be at present useful to the mathematical Student. I have already published the first three Sections in a separate form. I have here inserted, from the 6th Section, the construction for the place of a body in a parabolic orbit; from the 7th, the construction for the motion of a falling body for the two principal laws of force; the whole of the 9th Section, on Revolving Orbits; the 11th, on the Problems of Two and of Three Bodies; the 12th, on the Attraction of Spheres, as far as relates to the inverse square of the distance; and a proposition or two from the 13th Section. These are, I conceive, the propositions which have really that geometrical clearness and elegance which makes them convenient steps or illustrations in a Treatise on Dynamics; and so introduced, the reading of Newton may facilitate, and need not retard, the systematic study of the subject. But it is only thus that such reading is likely to be of service. If Newton's propositions are detached from the general body of Dynamics, made a separate subject and science, loaded with

deductions, comments and technicalities, they will certainly be far more of a hindrance than a help to us; and except they are treated in a purer spirit of geometry than is usually bestowed upon them in examinations, they will bring the young mathematician but small reward for the time and trouble he may employ upon them.

In presenting Newton's reasonings, I have in all cases substituted integrations for quadratures. All mathematicians will, I presume, allow that quadratures are merely integrations disguised. Such analytical processes, when they were new and strange, were very naturally translated into the synthetical language with which readers were then more familiar. But there is, in such contrivances, none of the peculiar evidence and reasoning of geometry: and if now, when neither writer nor reader thinks of finding the area of a curve otherwise than by an integral, we should still compel ourselves to represent an interval by an area, we should resemble Englishmen who torment themselves to talk to each other in a foreign language, uttered and understood with effort, instead of using the domestic phrases that come of themselves to the tip of the tongue. We may employ the diagram and the external forms of synthesis in such cases; but the thought and the whole idiom of our reasoning will inevitably be analytical.

I have therefore discarded, as utterly useless, and, in the present state of mathematics, absurd employments for the Student, those propositions in which the object is to reduce a problem to quadratures: for instance, the 39th proposition, where the areas, $ABFD$ and $ATVME$ are employed to obtain the velocity and the time: the 41st proposition, where we have the areas $ABFD$, $VDba$ and $VDca$ introduced, to determine the motion of a body in any trajectory: the 80th proposition, in which the area $ANB$ is used to find

the attraction of a sphere. I should indeed regret to see any of these ghosts of departed methods of calculation return to haunt the precincts of our examination halls; but this is now, I trust, a visionary fear.

For the same reason I have, in the propositions taken from the Third Book of the Principia, substituted integrations for the quadratures which occur in some instances. By this means the reasoning is materially abbreviated.

In the mode of presenting Newton's propositions I have gladly availed myself of Laplace's illustrations of some portions of the Lunar theory of Newton, given in the 5th volume of the Mécanique Céleste. Besides translating into analytical language Newton's processes for finding the *Variation* and other inequalities, Laplace has shewn that this mode of deducing the Variation might be generalised so as to give the differential equations between the radius vector, the angle of the Moon's orbit, and the time, which are the bases of the analytical investigation. But this speculation, curious as it is, I have omitted, not thinking it likely to be of any real use.

In expressing integrals, I have used the notation which has already been adopted in works published here by Professor Airy and others; according to which we omit the *differential* of the variable, (as $dx$), which in other books is introduced as a multiplier, and prefix to the *differential coefficient* the mark of integration, with a letter to indicate the variable quantity according to which the differentiation and integration are supposed to be performed (as $\int_x$). This notation appears to me an improvement on the older one. By using in all cases the differential coefficient instead of the differential, which usage thus becomes absolutely necessary, some superfluous steps and some sources of confusion are avoided. I have not, however, adopted the notation $d_x y$ instead of

$\frac{dy}{dx}$, for the differential coefficient of $y$ taken with regard to $x$; though this has been introduced by other Cambridge mathematicians, as a change co-ordinate with the above change in the notation of the integral. It did not seem to me that any point of practical convenience is secured by the second novelty, as is the case with the first: and having made the trial for some time, it appeared, so far as I could judge, that the application of the differential calculus to physical problems became, by the adoption of the new symbol, far less simple, clear and easy, than by the older plan.

Analytical as well as geometrical speculations may be unprofitable. The really important applications of mathematics are so numerous, that it is by no means desirable to employ the Student's time on detached and useless problems. I would point out, as coming under this description, the 5th, 6th, 7th, and 8th Sections of Chapter III. of the following Treatise. These Sections treat of the orbits described by the action of central forces varying inversely as the cube, as the 5th power, as the $n$th power; of the conditions of orbits which have asymptotic circles; and the like. Such problems have been treated by various mathematicians, (Cotes and Maclaurin in particular,) and it is perhaps right that they should be found in a treatise on this subject; but they are of no use in themselves; and I should be sorry that any examiner should give them an undue importance by making them the subjects of his questions.

There can be no doubt of the advantage which Students derive from working out the results of these and similar problems as exemplifications of the application of their principles. But those who would really use their mathematical acquirements for the improvement of their fellow Students in this place, may easily find better subjects for their skill. There are at present

a number of branches of natural science, to which mathematical calculation has been so far successfully applied, that they might form a portion of the subjects of study here, if we had clear and convenient treatises upon them.

I may mention, as examples of such subjects, the Theory of Magnetism as investigated by M. Poisson; Electro-dynamics according to the views of M. Ampere; the effects of Capillary Attraction as analyzed by Laplace and Poisson; the Theory of Waves as treated by Poisson and others; the Theory of Tides according to the method of Laplace. It appears, I think, that the most effectual mode of preparing such investigations for the reading of the elementary Student, is to break them into detached propositions, each of which is enunciated before it is proved. To shew that I am not unaware of the nature and difficulty of the operation which I thus recommend, I may mention that I have myself executed it with respect to M. Poisson's very beautiful theory of the Distribution of Electricity, in the mathematical part of the treatise on that subject in the Encyclopædia Metropolitana. By far the most valuable addition of this kind which could be made to our course, we fortunately already possess in Professor Airy's Tract on the Undulatory Theory of Light. But it ought to be the perpetual effort of all persons who take an interest in the progress of knowledge in this place, to bring the mathematical talent which is educed by our system of education to bear upon those questions which at present occupy the powers of the best mathematicians, and which offer us the hope of reducing successively a larger and larger portion of material nature under the dominion of known mathematical laws. Good treatises on such subjects as those above mentioned would have this tendency in an eminent degree.

On one of the subjects just mentioned I have dealt at some length in the following pages, I mean the Theory of the Tides;

following however here, as in the rest of my physical astronomy, the method of Newton. I have done this principally because we have not any published treatise on the subject to which the Student can be referred; and because the theory, though as yet far more incompletely compared with observation than any other consequence of the law of universal gravitation, is still one of great interest and importance. I should be very glad to see it more amply and successfully treated than I have been able to treat it. In what I have done on this subject I have availed myself of Mr Lubbock's valuable labours. The complete manner in which he has shewn the accordance of the tides at the port of London with the theory, is an important step in the subject; and his determination of the elements of the formulæ, as well as his general exposition of the requisite investigations, have been of great use to me.

Newton's greatest achievement, beyond doubt, was his inductive ascent from the complex cosmical phenomena which the universe presents, to the law of universal gravitation. This is generally acknowledged; yet the portion of the Principia in which the steps of this ascent are exhibited, the early part of the Third Book, seldom receives a very careful study, and scarcely ever appears in our examinations. And this is almost unavoidable; for we do not find, in this part of the work, that which is the appropriate business and peculiar pleasure of the mathematician; the deduction of consequences from general principles, known or granted. Instead of this, we have a collection of facts necessarily given by special enumeration and numerical record only: and they acquire that connexion and relation which makes them possible subjects of theoretical contemplation, only after they have been touched by a ray of that sagacity of genius by which general laws are revealed to

us. Before we can examine whether particular facts are rightly represented by a general law, the law itself must be enunciated; and this act at once assumes the facts as classed and combined by certain forms of dependence, whether this dependence be or be not truly stated in the enunciated law. This step once made, the comparison of such a law with numbers obtained from observation employs rather our arithmetical than our analytical skill, and has, very naturally, little attraction for the theoretical reader. Yet the inductive process has some features which may be dwelt on with profit. The laws thus verified may be afterwards included in more general laws, in the same manner as facts are in the more particular laws. The progress to the most general of physical laws, that of universal gravitation, includes several such successive ascents; and the exposition of the co-ordination and subordination of these steps, that is, the systematic statement of Newton's whole inductive reasoning, appears to me highly curious and instructive. I have therefore devoted a Chapter to this object, which is in general entirely neglected or slightly touched upon in works on Physics. I was the more desirous of doing this, inasmuch as some very eminent Continental mathematicians have recently exprest doubts whether the reasoning of Newton has really established the law of universal gravitation with complete accuracy and certainty. Nicolai, Bessel and Encke have seen reason to think that the observed effect of Jupiter upon the new planets, Juno, Pallas and Vesta, is different from the received estimate of the effect of the same planet upon Saturn; and have thus been led to ask whether the attraction be precisely proportional to the quantity of matter of the attracting body. It is perhaps more probable that the previous estimation of the effect of Jupiter on Saturn is erroneous, than that the law is so nearly true without being exactly so: but the mere

suspicion of such a possibility tends to give a peculiar interest to the examination of the proof hitherto accepted as decisive. In this part of the work I have availed myself of a very admirable Memoir of Bessel, in the Berlin Transactions for 1824. Its object is to shew that all the *broader* cosmical phenomena may be accounted for by other laws, as well as by that of universal gravitation; though it must be acknowledged that none of the other suppositions possess the simplicity of the Newtonian theory.

It follows from these views that the exact verification of Newton's law must depend upon careful examination of its *minuter* consequences; the perturbations of the planets, and the mutual attraction of terrestrial masses. I have given, as the last proposition in the work, the formulæ which are applicable to a peculiar experiment of the latter kind: the determination of the rate of a pendulum at a point below the earth's surface.

This experiment was suggested to me by Professor Airy. It is remarkable that it had been proposed at an earlier period, with somewhat of a similar view, by the anticipative mind of Bacon. In the Novum Organum, (Lib. II. XXXVI.) is this proposal of an *instantia crucis* to determine whether the tendency of bodies downwards arises from their being attracted by the earth as "a congregation of connatural bodies." "Fiat experimentum in profundis minerarum, utrum horologium non moveatur velocius quam solebat propter auctam virtutem ponderum. Quod si inveniatur virtus ponderum minui in sublimi, aggravari in subterraneis, recipiatur pro causâ ponderis attractio a massa corporea terræ."

In the experiment however to which the formulæ in Art. 139. refer, attraction as the cause of gravity was taken for granted, and the object was to determine the quantity of the attracting mass.

Attempts were made in 1826 and 1828 by Professor Airy and other gentlemen of this University, of whom I had the pleasure to be one, to carry into effect the experiment thus described at the mine of Dolcoath in Cornwall. Various obstacles prevented our obtaining a satisfactory result from the observations then instituted: but from all that I have seen and reflected, I retain a full persuasion that the experiment is one of those which are best adapted to determine the element sought, the mass of the earth; and it can hardly be said that all due efforts have been used to attain this determination, until the Dolcoath experiment has been properly repeated.

The present volume appears in the character of a portion of a new edition, although much the greater part of it is entirely new matter. As it has also a principle of unity, in its reference to the Physics of the universe, I have conceived that it would be more convenient to the reader that I should publish it immediately and separately. The remainder of the new edition of the 'Treatise on Dynamics,' which will contain the greatest portion of the analytical doctrine of the subject, will be prepared and published as soon as this can conveniently be done.

TRINITY COLLEGE, *April* 27, 1832.

# CONTENTS.

† *It is recommended to the Student to read Sections* V, VI, VII, VIII, *of Chap.* III, *as Examples only, and not as essential parts of the course.*

## CHAP. V.

### ON THE ATTRACTIONS OF BODIES.

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## PROPOSITIONS FROM THE PRINCIPIA,

### CONTAINED OR ILLUSTRATED IN THIS VOLUME.

*Principia*, Book I.

(Sect. 1, 2, 3 are given in the "*Introduction to Dynamics.*")

A

# TREATISE

ON

# DYNAMICS.

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1. THE object of the science of Dynamics is to determine, for any body or system of bodies, the motion which corresponds to any forces; and conversely, to determine the forces which correspond to any motion: that is, we have to investigate the relation of the time, space, velocity, and force, when bodies are in motion under given circumstances. In the "Introduction to Dynamics," we have proved that if $s$ be the space, described in the time $t$, in the direction of the motion, $v$ the velocity at the end of that time, $f$ the accelerating force in the direction of the motion, we have the equations $v = \frac{ds}{dt}$, $f = \frac{dv}{dt}$. These equations may be considered as the mathematical *definitions* of velocity and force.

Force, according to the common acceptation of the word, signifies the quality by which external causes produce an effect upon the motion of a body. We cannot conceive cause and effect without supposing certain principles which result from this relation; and by considering the manner in which these principles are exemplified in the phenomena of motion, we obtain Laws of Motion, (Introd. to Dynamics, p. 20.) by means of which Dynamical Problems may be reduced to equations.

*The first Law of Motion* is implied in the above-mentioned definition of force. *The second Law of Motion* is translated into analytical language in obtaining the equations of the curvilinear motion of a point. (Art. 19.)

*The third Law of Motion* gives us the connexion between the statical force requisite to produce equilibrium in a point, and the dynamical force by which its motion is affected.

When several points are connected, so that they must move or rest simultaneously, and one of them is acted on

by any pressure; this total pressure is to be resolved, on statical principles, into various partial pressures, acting on the several points; and these partial pressures must be so proportioned and adjusted that the points shall move in a manner consistently with their connexion. This rule, applied to determine the change of motion of a system, is an extension of the third law of motion to the case where the force is transferred from one part of the system to another by the connexion of the parts of the system. It is generally introduced into calculations by means of *D'Alembert's Principle.*

2. The distribution of the science of Dynamics according to the analytical conditions of the questions treated of, may be stated as follows.

The motion may be (*A*) that of a point, or (*B*) that of a rigid body: that is, the forces which act may either produce their whole effect immediately at the point acted on; or may require to be distributed through the system by D'Alembert's principle, in order that we may determine their effect.

(*A*) *The motion of a point* may be either (*a*) free, (*b*) constrained, or (*c*) resisted.

(*a*) The motion is *free*, when the forces do not depend upon the velocity or direction of the moving body; but are functions simply of the position of the body, either with regard to fixed points, or to some other body.

(*b*) The motion is *constrained*, when the moving body is subjected to the necessity of being always in a given line, or in a given surface, or at a given distance from another body, or to some similar condition. In this case, besides the external forces which act upon the body, and depend upon its position, it is acted on also by other forces, by means of which the conditions of its constrained motion are satisfied. Thus, when the body moves on a given line or surface, it is supposed to be acted upon by a force always perpendicular to the line or surface. And we have generally to eliminate this force in order to determine the motion.

(*c*) The motion is *resisted*, when the moving body, besides the forces which depend upon its position, is acted upon by

a force in the direction of its motion. This force, in general, depends upon the velocity of the moving body. By the introduction of the terms expressing this force into the equation, processes of integration different from those which are applicable to free motion become requisite.

The motion of a point may be constrained and resisted at the same time.

($B$) *The motion of a rigid body* may also be investigated as constrained and resisted; but these are conditions on which we shall not dwell.

3. In the case of the free and the unresisted constrained motion of points, there are certain properties and general propositions which are not true in the case of resisted motion.

Hence, the following is a convenient arrangement of the subject, and the one we shall follow.

Book I. The motion of points not resisted.

Part I. The free motion of points.

Part II. The constrained motion of points.

Book II. The motion of points resisted.

Book III. The motion of a rigid body.

In the present volume we shall, so far as the analytical theory is concerned, consider only the first Part of the first Book, the free motion of points. This will be divided into the investigation of the *rectilinear* motion of a point, (Chap. I.) and of the *curvilinear* motion of a point (Chap. II.). The case of curvilinear motion in which the force tends to a fixed *centre*, is of sufficient importance to be considered separately, (Chap. III.) and is subdivided into *nine* Sections, according to the conditions of force, velocity and direction by which the motion is determined. After this follows Chap. IV. on the motion of *several* points.

4. Some of the conclusions obtained in Capters III and IV, are so remarkably exemplified in some of the motions of the heavenly bodies, that it is proper to examine more closely the circumstances of these motions. It appears that the sun at-

tracts the planets, the earth and the moon, and that the earth attracts the moon, with forces which vary inversely as the square of the distance. The effects which these forces produce in the motions of the moon, give rise to a problem of great difficulty and complexity: (the *problem of the three bodies.*) And though its exact solution is impossible, and the systematic approximations to such a solution are beyond the scope of a work like the present, it seemed desirable to present to the reader certain modes of approximation to some of the leading circumstances of the motion, which Newton applied to this problem with very considerable success, and which do not involve the difficulties of the systematic analytical solution.

5. The motions of the sun, planets, earth and moon, are deduced by supposing these bodies to be *points*, each of which exerts an attraction varying inversely as the square of the distance. But it appears that if, instead of conceiving one such force to tend to the center of the earth, we conceive forces, acting according to the same Law, to tend to every point of the earth's mass, the result will be precisely the same. The *attraction* of the sphere will in this case vary according to the same law as the attraction of the point. The proof of this and similar propositions concerning attractions, forms the subject of the two first sections of Chapter v: two other sections are employed in the consideration of the application of those principles to certain terrestrial phenomena, viz. the Figure of the Earth, and the Tides.

6. In the preceding part of the volume it has been supposed, that the different bodies of the universe exercise upon one another forces which are inversely as the square of the distance; that is, the law of universal gravitation has been assumed: and by reasoning *deductively* from this *assumption*, various results have been obtained, which can be compared with the facts as they occur in nature. But, in order that our knowledge may be shewn to be a true theory of the facts, it must appear, that beginning with the facts, and reasoning *inductively* from these, we arrive at the law of universal gravitation, and at no other. To point out the steps of this inductive ascent, is the object of Chapter VI.

# BOOK I.

## THE MOTION OF POINTS IN A NON-RESISTING SPACE.

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# PART I.

## THE FREE MOTION OF POINTS.

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# CHAP. I.

## THE RECTILINEAR MOTION OF A POINT.

7. WHEN a point moves freely in a straight line, this line must be that in which the force acts. In this case we can immediately apply the equations

$$v = \frac{ds}{dt}, \quad f = \frac{dv}{dt};$$

where $t$, $s$, $v$, are the time of motion, space described, and velocity of a body, which is acted on by an accelerating force $f$ in the direction of its motion.

These equations would enable us immediately to obtain finite relations among the quantities in question, in several cases. For instance, if the velocity were given in terms of the time, we could find the space described, by integrating the first equation; and if the force were known in terms of the time, we might in the same manner obtain the velocity from the second equation. In general, however, the force depends upon the position of the body.

8. PROPOSITION. *The force being supposed to depend upon the body's position, it is required to find the velocity at any point of the motion, and the time of describing any space.*

We have $\quad f = \dfrac{dv}{dt} = \dfrac{dv}{ds} \cdot \dfrac{ds}{dt} = v\dfrac{dv}{ds}.$

If $f$ be a function of $s$, we may now integate both sides with regard to $s$; and using $\int_s$ as the symbol of this integration, and observing that the integral of $v\frac{dv}{ds}$ is $\frac{1}{2}v^2$, we have
$$\tfrac{1}{2}v^2 = \int_s f, \quad v^2 = 2\int_s f;$$
when $v$ is known in terms of $s$.

We can then determine $t$ in terms of $s$; for
$$\frac{ds}{dt} = v = v\frac{dt}{ds}\cdot\frac{ds}{dt}; \text{ whence } 1 = v\cdot\frac{dt}{ds}; \text{ and } \frac{1}{v} = \frac{dt}{ds}.$$

Integrating with regard to $s$, $t = \int_s \frac{1}{v}$.

A constant quantity will be introduced in each integration.

9. If the force be a function of the body's distance from a point *towards* which it is moving, let $x$ be this distance. Then $x$ decreases as $t$ increases, and $\frac{dx}{dt}$ will be negative; therefore, in order that $v$ may be positive, we have
$$v = -\frac{dx}{dt}, \quad f = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = -v\frac{dv}{dx}:$$
$$v^2 = -2\int_x f.$$
$$\text{Also,} \quad -\frac{dx}{dt} = v = v\frac{dt}{dx}\cdot\frac{dx}{dt}; \quad \text{whence} \quad -\frac{1}{v} = \frac{dt}{dx}:$$
$$t = -\int_x \frac{1}{v}.$$

10. We now proceed to the calculation for different suppositions of force.

One of the most common and useful suppositions is, that the force tends to a certain point or centre, and varies according to some direct or inverse power of the distance from this centre. Let $S$, fig. 1, be the centre of force; $SP = x$; and when a point is at $P$, let it be urged towards $S$ by a

force $\frac{\mu}{x^n}$, or $\mu x^n$, $\mu$ being a constant quantity. The quantity $\mu$ depends upon the attractive power residing in $S$, and is called the *absolute force.* It is measured by the accelerating force at the distance 1; for, making $x = 1$, we have the force $= \mu$.

It is supposed that $\frac{\mu}{x^n}$ or $\mu x^n$ expresses the accelerating force on a particle $P$, whatever be the magnitude of the particle: the moving force or pressure produced in $P$ by the attraction of $S$ is greater as the mass acted on is greater.

11. PROP. *A body* P *falls from rest from a given point* A, *fig.* 1, *towards a centre of force* S, *varying as some power of the distance* SP: *it is required to determine the motion**.

Let $SA = a$, $SP = x$; and take the force to be as some *inverse* power of the distance, and $= \frac{\mu}{x^n}$.

1st, To find the velocity:—by Art. 9,

$$v^2 = -2\int_x \frac{\mu}{x^n} = \frac{2\mu}{(n-1)x^{n-1}} + C,$$

$C$ being an arbitrary constant quantity, to be determined.

When $x = a$, $v = 0$; $\therefore v^2 = \frac{2\mu}{n-1}\left\{\frac{1}{x^{n-1}} - \frac{1}{a^{n-1}}\right\}$

$$v = \frac{(2\mu)^{\frac{1}{2}}}{(n-1)^{\frac{1}{2}}}\left(\frac{a^{n-1} - x^{n-1}}{a^{n-1}x^{n-1}}\right)^{\frac{1}{2}}.$$

This gives the velocity when $n > 1$.

If $n = 1$ this integration fails, and recurring to the differential expression, we have

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* This is Prop. 39, Book I. of the *Principia*.

'To determine the motion,' "definire motum," implies the problems of obtaining the relation of the space, velocity, and time in finite terms, that is, freed from differentials.

$$v^2 = -2\int_x \frac{\mu}{x} = -2\mu\,\mathrm{l}\,x + C = 2\mu\,\mathrm{l}\,\frac{a}{x}.$$

If $n<1$, $v^2 = \dfrac{2\mu}{1-n}(a^{1-n}x^{1-n})$.

If the force vary as some *direct* power of the distance, let $f = \mu x^n$, and we have

$$v^2 = \frac{2\mu}{n+1}(a^{n+1} - x^{n+1}).$$

In cases where the force varies as some inverse power $n$, greater than 1, when $x = 0$, $v = inf.$ or the velocity of falling to the centre is infinite.

In the same case; when $a = inf.$ $v$ remains finite, and

$$v^2 = \frac{2\mu}{(n-1)x^{n-1}};$$

or the velocity of falling from an infinite distance to the distance $x$ is finite.

If the force vary inversely as the distance, both these velocities are infinite.

In all other cases, the velocity from an infinite distance to a finite one, is infinite; and the velocity from a finite distance to the centre, is finite.

If the body, instead of falling *from rest* at a distance $a$, be *projected* upwards or downwards with a velocity $V$, we have, when $x = a$, $v = V$, if the body be projected in the direction of the force, and $v = -V$, if it be projected in the opposite direction.

In both these cases, $v^2 = V^2 + \dfrac{2\mu}{n-1}\left(\dfrac{1}{x^{n-1}} - \dfrac{1}{a^{n-1}}\right)$.

2d, To find the time;

$$t = -\int_x \frac{1}{v} = -\int_x \frac{1}{\dfrac{(2\mu)^{\frac{1}{2}}}{(n-1)^{\frac{1}{2}}}\left(\dfrac{a^{n-1} - x^{n-1}}{a^{n-1}x^{n-1}}\right)^{\frac{1}{2}}}$$

$$= -\frac{(n-1)^{\frac{1}{2}} \cdot a^{\frac{n-1}{2}}}{(2\mu)^{\frac{1}{2}}}\int_x \frac{x^{\frac{n-1}{2}}}{(a^{n-1} - x^{n-1})^{\frac{1}{2}}},$$

which can be integrated only in particular cases: see *Lacroix*, Elem. Treat. Art. 169.

1st, We can integrate if $\frac{m'}{n'}$ be a whole number, where

$$m' - 1 = \frac{n-1}{2} \text{ and } n' = n - 1;$$

that is, calling $r$ a whole number, if

$$\frac{\frac{n-1}{2}+1}{n-1} = r, \text{ or } n + 1 = 2rn - 2r; \quad \therefore n = \frac{2r+1}{2r-1};$$

this comprehends the cases $n = -1, \frac{1}{3}, \frac{3}{5}, \frac{5}{7}$, &c.

also $n = \frac{3}{1}, \frac{5}{3}, \frac{7}{5}$, &c.

2d, We can integrate if $\frac{m'}{n'} - \frac{1}{2}$ be a whole number; suppose

$$\frac{m'}{n'} - \frac{1}{2} \text{ or } \frac{n+1}{2(n-1)} - \frac{1}{2} = r; \quad \therefore \frac{1}{n-1} = r; \quad \therefore n = \frac{r+1}{r},$$

this comprehends the cases $n = 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$, &c.

also $n = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$, &c.

Hence the only laws of force expressed by integral powers, for which we can find the time, are

$$\text{force} \propto \text{const., force} \propto \text{dist., force} \propto \frac{1}{(\text{dist.})^2}, \text{ force} \propto \frac{1}{(\text{dist.})^3}.$$

The most simple fractional powers are

$$\text{force} \propto \frac{1}{(\text{dist.})^{\frac{1}{2}}}, \text{ force} \propto \frac{1}{(\text{dist.})^{\frac{1}{3}}}, \text{ force} \propto \frac{1}{(\text{dist.})^{\frac{3}{2}}}.$$

If the force be repulsive, the process of finding the velocity and time will be the same as above, except that the signs will be different.

In that case, if force $= \frac{\mu}{x^n}$,

$$v^2 = 2\int_s \frac{\mu}{x^n}, \quad t = \int_s \frac{1}{v}.$$

12. In many of the integrable cases, it is better to employ particular methods, than the general substitution for making the differential expressions rational.

Ex. 1. The force varies directly as the distance:

$$f = \mu x;$$

$$\therefore\ v^2 = -2\int_s \mu x = C - \mu x^2 = \mu(a^2 - x^2),$$

$$v = \mu^{\frac{1}{2}}(a^2 - x^2)^{\frac{1}{2}};$$

$$t = -\int_s \frac{1}{v} = -\int_s \frac{1}{\mu^{\frac{1}{2}}(a^2 - x^2)^{\frac{1}{2}}},$$

$$t = \frac{1}{\mu^{\frac{1}{2}}} \cos^{-1}\frac{x}{a} + C;$$

and if $t$ begin when $x = a$,

$$t = \frac{1}{\mu^{\frac{1}{2}}} \cos^{-1}\frac{x}{a}.$$

Cor. If with radius $SA = a$, fig. 2, a quadrant be described, and $PQ$ be perpendicular to $SA$; and if $SP = x$: $AQ =$ arc whose cos. is $x$ and rad. $a = a \times$ arc whose cos. is $\frac{x}{a}$ and rad. 1;

$$\text{velocity at } P = \mu^{\frac{1}{2}} . PQ,$$

$$\text{time in } AP = \frac{\text{arc } AQ}{\mu^{\frac{1}{2}} a} = \frac{\text{arc } AQ}{\text{vel. at } S}.$$

We have for the whole time of falling to the centre, making $x = 0$; $t = \frac{\pi}{2\mu^{\frac{1}{2}}}$.

Ex. 2. The force is constant:

$v^2 = 2\int_s f$; $\therefore v^2 = 2fs$; the motion beginning when $s = 0$.

$$t = \int_s \frac{1}{\sqrt{(2fs)}}; \quad \therefore t = \sqrt{\frac{2s}{f}}; \ t \text{ being } 0 \text{ when } s \text{ is } 0.$$

If the constant force be gravity, represented by $g$,

$$v^2 = 2gs, \text{ and } t = \sqrt{\frac{2s}{g}}, \text{ or } s = \tfrac{1}{2}gt^2.$$

Ex. 3. The force varies inversely as the square of the distance,

$$f = \frac{\mu}{x^2}, \quad v^2 = -2\int_s \frac{\mu}{x^2} = 2\mu\left(\frac{1}{x} - \frac{1}{a}\right):$$

$$t = -\int_s \frac{a^{\frac{1}{2}} x^{\frac{1}{2}}}{(2\mu)^{\frac{1}{2}}(a-x)^{\frac{1}{2}}} = -\left(\frac{a}{2\mu}\right)^{\frac{1}{2}} \int_s \frac{x}{(ax - x^2)^{\frac{1}{2}}}.$$

$$t = \left(\frac{a}{2\mu}\right)^{\frac{1}{2}} \left\{(ax - x^2)^{\frac{1}{2}} - \frac{a}{2}\,\text{versin}^{-1}\frac{2x}{a}\right\} + C;$$

and, $t$ being supposed to begin when $x = a$, since $\text{versin}^{-1} 2 = \pi$,

$$t = \left(\frac{a}{2\mu}\right)^{\frac{1}{2}} \left\{(ax - x^2)^{\frac{1}{2}} + \frac{a}{2}\left(\pi - \text{versin}^{-1}\frac{2x}{a}\right)\right\}.$$

We have for the whole time of falling to the centre, making $x = 0$, $t = \dfrac{\pi a^{\frac{3}{2}}}{2(2\mu)^{\frac{1}{2}}}$.

Cor. On $AS = a$, fig. 3, let a semicircle be described, with centre $C$; and let $PQ$ be drawn perpendicular to $AS$ meeting it,

$$PQ = \sqrt{(SP \cdot PA)} = \sqrt{(ax - x^2)}; \quad \text{arc } AQS = \frac{\pi a}{2}.$$

$$\text{arc } SQ = SC \times \text{ang. } SCQ = SC \times \text{ang. whose ver. sin. is } \frac{SP}{SC}$$

$$= \frac{a}{2}\,\text{versin}^{-1}\frac{2x}{a}.$$

Hence, time in $AP = \left(\dfrac{a}{2\mu}\right)^{\frac{1}{2}} (PQ + \text{arc } AQS - \text{arc } SQ)$

$$= \left(\frac{a}{2\mu}\right)^{\frac{1}{2}} (PQ + \text{arc } AQ).$$

$$\text{Hence, also time} = \left(\frac{a}{2\mu}\right)^{\frac{1}{2}} \left(\frac{PQ.\frac{1}{2}SC + \text{arc } AQ.\frac{1}{2}AC}{\frac{1}{2}AC}\right)$$

$$= \frac{2\sqrt{2}}{(a\mu)^{\frac{1}{2}}} \text{area } ASQ^*.$$

If $SQ$ be produced to meet in $R$ a tangent to the semicircle at $A$;

$$AR = SA.\frac{PQ}{PS} = \frac{a\sqrt{(ax - x^2)}}{x} = a^{\frac{3}{2}}\left(\frac{1}{x} - \frac{1}{a}\right)^{\frac{1}{2}};$$

$$\therefore\ v = \frac{(2\mu)^{\frac{1}{2}}}{a^{\frac{3}{2}}}.AR.$$

Ex. 4. The force varies inversely as the cube of the distance;

$$v = \sqrt{\mu}.\frac{\sqrt{(a^2 - x^2)}}{ax}; \quad t = a.\frac{\sqrt{(a^2 - x^2)}}{\sqrt{\mu}}.$$

Cor. If with centre $S$ and radius $SA$, fig. 4, we describe a circle, and make $PQ$, $AR$ perpendicular to $SA$, and draw $SQR$;

$$v = \frac{\sqrt{\mu}}{a^2}.AR; \quad t = \frac{a}{\sqrt{\mu}}.PQ.$$

Ex. 5. The force varies inversely as the square of the distance, and is repulsive:

$$v = \sqrt{(2\mu)}\left\{\frac{1}{a} - \frac{1}{x}\right\}^{\frac{1}{2}}: \quad t = \left(\frac{a}{2\mu}\right)^{\frac{1}{2}} \int_a \frac{x}{\sqrt{(x^2 - ax)}}.$$

Cor. If with focus $S$, fig. 5, and vertex $A$, the point from which the body begins to move, we describe a parabola, and take $SQ = SP$; $SY$ being the perpendicular upon the tangent $QY$, we have

$$\frac{d.\text{arc } AQ}{d.SQ} = \frac{SQ}{QY} = \frac{SQ}{\sqrt{(SQ^2 - SY^2)}}$$

$$= \frac{SQ}{\sqrt{(SQ^2 - SQ.SA)}} = \frac{x}{\sqrt{(x^2 - ax)}};$$

* *Principia*, Book I. Prop. 32.

$$t = \left(\frac{a}{2\mu}\right)^{\frac{1}{2}} \int_x \frac{d \,.\, \text{arc } AQ}{d \,.\, x} = \left(\frac{a}{2\mu}\right)^{\frac{1}{2}} AQ.$$

Ex. 6. The force varies inversely as the square root of the distance:

$$v = 2\mu^{\frac{1}{2}} \,.\, \left\{a^{\frac{1}{2}} - x^{\frac{1}{2}}\right\}^{\frac{1}{2}};$$

$$t = \frac{2}{3\mu^{\frac{1}{2}}} \,.\, \left\{x^{\frac{1}{2}} + 2a^{\frac{1}{2}}\right\} \,.\, \left\{a^{\frac{1}{2}} - x^{\frac{1}{2}}\right\}^{\frac{1}{2}}.$$

Geometrical Investigation *of Examples* 1 *and* 3.

The velocity and time in these cases may be obtained without the use of the Differential Calculus, by means of the Propositions contained in the "Introduction."

(Newton, Book I. Prop. xxxviii.)

13. Prop. *The force varying directly as the distance from the centre, the times of falling, the velocities acquired, and the spaces described, are as the arcs, sines and versed sines respectively, in a circle of which the radius is the original distance.*

Let $SA$, fig. 145, be the original distance, $AE$ a quadrant with radius $SA$. Let $APB$ be an ellipse, of which the semi-axis major is $SA$; and let a body describe this ellipse, and another body describe the circle $AE$ by the same force at $S$. Then, by Prop. x. Cor. 2, of the Introduction, the periodic times in these orbits will be equal, and therefore the times of describing the quadrants $AB$, $AE$, which are each $\frac{1}{4}$ of the periodic time, will be equal.

Let $CPD$ be perpendicular to $SA$; then

$$\frac{\text{time in } AP}{\text{time in } AB} = \frac{\text{area } ASP}{\text{area } ASB} = \frac{\text{area } ASD}{\text{area } ASE} = \frac{\text{time in } AD}{\text{time in } AE}.$$

Hence, time in $AP$ = time in $AD$; and if the motions begin together at $A$, when the body in the circle is at $D$ the body in the ellipse will be at $P$, whatever be the semi-axis minor $SB$.

Therefore this will be true when the semi-axis $SB$ vanishes, in which case the motion of the body in the ellipse becomes the rectilinear motion of a body falling in the line $AS$. And when this body is at $S$, the velocity is equal and parallel to that at $E$.

But the motion of the body in the circle is uniform: hence

$$\text{time in } AC = \text{time in } AD = \frac{AD}{\text{vel. at } E} = \frac{AD}{\text{vel. at } S},$$

as already proved in page 10.

Also for the velocity at $C$; by Newton's Principia, Sect. 2. Prop. VI. (Cor. 5 in the "Introduction to Dynamics.")

$$\text{Vel.}^2 \text{ at } P = 2 \text{ force} . \tfrac{1}{4} \text{ chord of curv.} = \mu . SP . \frac{SQ^2}{SP},$$

$SQ$ being the semidiameter conjugate to $SP$. Therefore vel.$^2$ at $P = \mu . SQ^2$. But $SP^2 + SQ^2 = AS^2 + BS^2$ by Conics.

And when $BS$ vanishes, $SP^2 + SQ^2 = AS^2$;

therefore in that case $SQ^2 = SA^2 - SC^2 = SD^2 - SC^2 = CD^2$:

and velocity at $C = \mu^{\frac{1}{2}} . CD$.

Hence the space described being the versed sine $AC$, the time is as the arc $AD$, and the velocity as the sine $CD$.

(Newton, Book I. Prop. XXXIII.)

14. Prop. *The force varying inversely as the square of the distance, a body falls in a straight line towards the centre, from* A *to* C; *then*

*vel. at* C : *vel. in circle with rad.* BC :: $\sqrt{\text{AC}} : \sqrt{\tfrac{1}{2}\text{AB}}$.

Let a body move in the ellipse $APB$, fig. 146, about the focus $S$: and let another body describe a circle with radius $SP$. Then by Newton, Prop. VI.,

$$\frac{\text{vel.}^2 \text{ in ellipse } AP}{\text{vel.}^2 \text{ in circle with rad. } SP} = \frac{\text{chord curvature at } P \text{ through } S}{2SP}$$

$$= \frac{2SP . HP}{AO . 2SP} = \frac{HP}{AO}.$$

Let the minor axis of the ellipse $APB$ be diminished without limit: then the motion in the ellipse becomes the rectilinear motion of a falling body, and $P$ coincides with $C$, and $S$ with $B$.

$$\frac{\text{vel. of falling body at } C}{\text{vel. in circle with rad. } BC} = \frac{\sqrt{AC}}{\sqrt{AO}}.$$

Cor. 1. The velocity in the circle is that which will be acquired by falling through an external space equal to the radius; for when $C$ is at $O$, velocity of falling = velocity in the circle.

Cor. 2. If the force be $= \frac{\mu}{SP^2}$,

the velocity in circle with rad. $BC = \sqrt{\frac{2\mu}{BC^2} \cdot \frac{BC}{2}} = \sqrt{\frac{\mu}{BC}}$.

Hence the velocity of a falling body at $C$

$$= \sqrt{\frac{\mu \cdot AC}{BC \cdot AO}} = \sqrt{\frac{\mu \cdot AC \cdot BC}{AO \cdot BC^2}} = \sqrt{\frac{\mu \cdot CD^2}{AO \cdot BC^2}}$$

$$= \sqrt{\frac{2\mu}{AB}} \cdot \frac{CD}{BC} = \sqrt{\frac{2\mu}{AB}}\,\frac{AE}{AB},$$

agreeing with page 12.

(Newton, Book I. Prop. xxxv.)

15. Prop. *In the same case, to find the time of describing any portion* AC *of the descent.*

Let a body move in the ellipse $APB$ about the focus $S$. Then we have

$$\frac{\text{time in } AP}{\text{periodic time in the ellipse}} = \frac{\text{area } ASP}{\text{area of the ellipse}} = \frac{\text{area } ASD}{\text{area of the circle}}.$$

And the periodic time in the ellipse is equal to that in a circle on the same semidiameter.

Hence time in $AP = \dfrac{\text{periodic time in the circle}}{\text{area of circle}} \cdot \text{area } ASD$

$$= \frac{\text{area } ASD}{\text{area in time 1 in the circle}}.$$

Now the velocity in the circle $= \sqrt{\dfrac{2\mu}{BO^2} \cdot \dfrac{BO}{2}} = \sqrt{\dfrac{2\mu}{AB}}$.

Hence the area in the time 1 $= \frac{1}{2} \cdot BO \sqrt{\dfrac{2\mu}{AB}} = \dfrac{\sqrt{2\mu \cdot AB}}{4}$

$$\text{and time in } ASP = \frac{4 \text{ area } ASD}{\sqrt{2\mu \cdot AB}}.$$

When the minor axis of the ellipse is diminished indefinitely, the motion becomes that of a falling body. But in this case $P$ coincides with $C$ and $S$ with $B$. Therefore

$$\text{time of falling through } AC = \frac{4}{\sqrt{2\mu \cdot AB}} \text{ area } ABD.$$

This agrees with what we have proved in page 12.

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16. Prop. *A body acted upon by a force varying as any power of the distance, falls to the centre from a given distance* (a): *to find the whole time of falling to the centre.*

By Art. 11, we have, if force $= \dfrac{\mu}{x^n}$,

$$\frac{dt}{dx} = -\frac{(n-1)^{\frac{1}{2}} a^{\frac{n-1}{2}}}{(2\mu)^{\frac{1}{2}}} \cdot \frac{x^{\frac{n-1}{2}}}{\{a^{n-1} - x^{n-1}\}^{\frac{1}{2}}},$$

$$= -\frac{(n-1)^{\frac{1}{2}} x^{\frac{n-1}{2}}}{(2\mu)^{\frac{1}{2}}} \cdot \left\{1 - \frac{x^{n-1}}{a^{n-1}}\right\}^{-\frac{1}{2}},$$

$$= -\frac{(n-1)^{\frac{1}{2}} x^{\frac{n-1}{2}}}{(2\mu)^{\frac{1}{2}}} \cdot \left\{1 + \frac{1}{2} \cdot \frac{x^{n-1}}{a^{n-1}} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^{2n-2}}{a^{2n-1}} + \&c.\right\},$$

Multiplying and integrating, we shall have

$$t = \frac{(n-1)^{\frac{1}{2}}}{(2\mu)^{\frac{1}{2}}}\left\{C - \frac{2}{n+1}\cdot x^{\frac{n+1}{2}} - \frac{1}{2}\cdot\frac{2}{3n-1}\cdot\frac{x^{\frac{3n-1}{2}}}{a^{n-1}}\right.$$
$$\left. - \frac{1\,.\,3}{2\,.\,4}\cdot\frac{2}{5n-3}\cdot\frac{x^{\frac{5n-3}{2}}}{a^{2n-2}} - \&c.\right\}:$$

and this, taken from $x = a$, to $x = 0$, gives for the whole time

$$t = \frac{(n-1)^{\frac{1}{2}}}{(2\mu)^{\frac{1}{2}}}\cdot 2a^{\frac{n+1}{2}}\left\{\frac{1}{n+1} + \frac{1}{2}\cdot\frac{1}{3n-1}\right.$$
$$\left. + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{5n-3} + \&c.\right\}.$$

Cor. 1. From different distances the times of falling to the same centre as $a^{\frac{n+1}{2}}$.

Hence if force $\propto$ dist. time $\propto 1$, or is constant,

if force $\propto 1$, or is constant, time $\propto \sqrt{(\text{dist.})}$

if force $\propto \dfrac{1}{(\text{dist.})}$, time $\propto$ dist.

if force $\propto \dfrac{1}{(\text{dist.})^2}$, time $\propto (\text{dist.})^{\frac{3}{2}}$,

if force $\propto \dfrac{1}{(\text{dist.})^3}$, time $\propto (\text{dist.})^2$,

if force $\propto \dfrac{1}{(\text{dist.})^{\frac{1}{2}}}$, time $\propto (\text{dist.})^{\frac{3}{4}}$.

Cor. 2. In all these cases the time is greater (or at least not less) as the distance is greater; but if the force vary in a higher direct ratio than the simple power of the distance, the contrary will be the case.

Thus, if force $\propto (\text{dist.})^2$, time $\propto \dfrac{1}{(\text{dist.})^{\frac{1}{2}}}$,

if force $\propto (\text{dist.})^3$, time $\propto \dfrac{1}{(\text{dist.})}$.

COR. 3. The integration for finding the time when the force is inversely as the distance, is not properly included in this case, and is considered in the following problem.

17. PROB. *When the force varies inversely as the distance, to find the whole time of the descent to the centre.*

Let any distance $SP$, fig. 1, $= r$; hence,

$$f = -\frac{\mu}{r}, \quad v^2 = -2\int_r \frac{\mu}{r} = 2\mu\,\mathrm{l}\frac{a}{r},$$

$$\frac{dt}{dr} = -\frac{1}{v} = -\frac{1}{(2\mu)^{\frac{1}{2}}}\,\frac{1}{\sqrt{\mathrm{l}\frac{a}{r}}}.$$

And our object must now be to integrate this expression from $r = a$, to $r = 0$.

Let $\sqrt{\mathrm{l}\frac{a}{r}} = x$; $\therefore \mathrm{l}\frac{a}{r} = x^2$, $\frac{a}{r} = e^{x^2}$, $r = \frac{a}{e^{x^2}}$, $\frac{dr}{dx} = -\frac{2ax}{e^{x^2}}$;

$$\frac{dt}{dx} = \frac{dt}{dr}\cdot\frac{dr}{dx} = \frac{2a}{(2\mu)^{\frac{1}{2}}}\cdot e^{-x^2},$$

$$\text{and } t = \frac{2a}{(2\mu)^{\frac{1}{2}}}\int_x e^{-x^2}$$

from $x = 0$ to $x = \infty$.

Now let there be a curve $BQ$, fig. 6, of which the ordinates are $CO = u$, $OQ = z$: and let its equation be $z = e^{-u^2}$. Let this curve revolve round the axis $CB$, parallel to $z$, through a quadrant, so as to generate the surface $BQQ'$. We may find the solid content thus generated by supposing the plane $CBPN$ to revolve through an angle $\delta\theta = NCn$. If $CN = u$, we shall have $Nn = u\delta\theta$, and if we take a portion of the triangle whose breadth along $CN$ is $\delta u$, and conceive, standing upon it, a prism whose height is $NP$ or $z$, the solid content of this prism will ultimately be $z\,.\,u\delta\theta\,.\,\delta u$. Hence the differential coefficient of the wedge $BCPn$ with regard to $u$, will be the limit of the ratio of the increment of the wedge to the increment of $u$, that is, it will be $zu\delta\theta$: and the wedge $= \delta\theta\int_u zu = \delta\theta\int_u e^{-u^2}u$, taken from $C$ to $N$.

And the solid content of the figure when the plane has revolved through a quadrant, will manifestly be

$$\frac{\pi}{2}\int_u e^{-u^2}u = \frac{\pi}{4}\cdot\left\{C - e^{-u^2}\right\};$$

if this be taken from $C$, when $u = 0$, integral $= 0$;

$$\text{hence, solid content} = \frac{\pi}{4}\cdot\left\{1 - e^{-u^2}\right\}.$$

And if we suppose the solid to be extended to infinity, so as to comprehend the whole space between the planes $BCX$, $BCY$, and the curve surface, we must make $u$ infinite, and we have the solid content $= \frac{\pi}{4}$.

But we may find this solid content in another manner, by referring the surface to three rectangular co-ordinates, $CM = x$, $MN = y$, $NP = z$: and it will then be equal to $\int_x\int_y z$. (*Lacroix*, Elem. Treat. Art. 247.)

Now $u^2 = CN^2 = x^2 + y^2$, and $z = e^{-u^2} = e^{-(x^2+y^2)}$.

$$\begin{aligned}\text{Hence, content } &= \int_x\int_y e^{-x^2-y^2}\\ &= \int_y\int_x e^{-x^2}\cdot e^{-y^2}\\ &= \int_x e^{-x^2}\cdot\int_y e^{-y^2};\end{aligned}$$

because in integrating with respect to $y$, $x$ may be considered as constant.

And for the whole content we must take the integrals from $x = 0$ to $x = \infty$, and from $y = 0$ to $y = \infty$; and in this case, $\int_x e^{-x^2}$ and $\int_y e^{-y^2}$ will manifestly be equal. Hence, whole content $= (\int_x e^{-x^2})^2$, from $x = 0$ to $x = \infty$;

$$\therefore\ \frac{\pi}{4} = \left(\int_x e^{-x^2}\right)^2, \text{ from } x = 0 \text{ to } x = \infty,$$

$$\frac{\sqrt{\pi}}{2} = \int_x e^{-x^2}, \text{ from } x = 0 \text{ to } x = \infty.$$

And hence time to centre in this Prop. $= \dfrac{a\sqrt{\pi}}{\sqrt{2\mu}}$.

# CHAP. II.

## THE CURVILINEAR MOTION OF A POINT.

18. WHEN a point in motion is acted on by a force which is not in the direction of its motion, it will be perpetually deflected from its path, so as to describe a curve line. The quantity of this deflection will be regulated by the second law of motion, as explained in the Introduction, p. 23. By that law it is asserted, that if a point at $P$ be moving with a velocity which would, in a given time, carry it through the space $PR$, fig. 7; and if, during its motion, it be acted on by a constant force, always parallel to itself, which would, in the same time, make it move through a space $Pp$ from rest, it will be found, at the end of that time, in a point $r$, determined by completing the parallelogram $Rp$.

If the force which acts upon the body, be variable in magnitude, or direction, or both, we can no longer in the same manner find the place of the body at the end of a *finite* time from $P$. The second law of motion is then applicable *ultimately* only; that is, to the motion of the body during an indefinitely small time; as explained in the Introduction, Cor. 1, to Law II.

19. PROP. *To find the equations of motion of a body, moving in a plane and acted upon by any forces in that plane.*

Fig. 7. Let $t$ be the time from a given epoch, till the body arrives at $P$, and $t+h$ till it arrives at $Q$, so that $h$ is the time of motion in $PQ$. Also let $AM$, $MP$ be rectangular co-ordinates to the point $P$, and be called $x$ and $y$: similarly, let $AN$, $NQ$, and $AO$, $OR$, be co-ordinates parallel to these; $PI$ and $RK$ parallel to $AN$. Let the force at $P$ be called $P$, and let the angle which it makes with $x$,

be called $\alpha$. Also let the velocity at $P$ be called $V$, and let the angle which it makes with $x$ be called $\theta$.

Let $PR$ be described in the time $h$, with the velocity at $P$, and let $Rr$ be the space through which a point would in the same time be described by the force at $P$. We shall then have

$$PR = Vh: \text{ and } Rr = \tfrac{1}{2} Ph^2;$$

because $Rr$ is described by a constant force. (Ch. I. Ex. 2.)

Hence, $PH = Vh \cos. \theta$; $RH = Vh \sin. \theta$.

Also if $Rs$, $sr$ be parallel to $AM$, $MP$,

$$Rs = \tfrac{1}{2} Ph^2 \cos. \alpha; \quad sr = \tfrac{1}{2} Ph^2 \sin. \alpha.$$

But by Taylor's Theorem, considering $x$ and $y$ as functions of $t$, and $t$ as the independent variable,

$$AN = x + \frac{dx}{dt} . h + \frac{d^2 x}{dt^2} . \frac{h^2}{1 . 2} + \&c.$$

$$NQ = y + \frac{dy}{dt} . h + \frac{d^2 y}{dt^2} . \frac{h^2}{1 . 2} + \&c.$$

Hence,

$$RK = MN - PH = \frac{dx}{dt} . h + \frac{d^2 x}{dt^2} . \frac{h^2}{1 . 2} + \&c. - Vh \cos. \theta.$$

$$KQ = IQ - RH = \frac{dy}{dt} . h + \frac{d^2 y}{dt^2} . \frac{h^2}{1 . 2} + \&c. - Vh \sin. \theta.$$

Now, by last article since $Rr$ ultimately coincides with $RQ$, we have ultimately $Rs$, $RK$ equal, and also $sr$, $KQ$. Hence, ultimately

$$\left(\frac{dx}{dt} - V \cos. \theta\right) h + \frac{d^2 x}{dt^2} . \frac{h^2}{1 . 2} + \&c. = \tfrac{1}{2} Ph^2 . \cos. \alpha.$$

$$\left(\frac{dy}{dt} - V \sin. \theta\right) h + \frac{d^2 y}{dt^2} . \frac{h^2}{1 . 2} + \&c. = \tfrac{1}{2} Ph^2 . \sin. \alpha.$$

Whence we must necessarily have, equating coefficients of $h$,

$$\frac{dx}{dt} - V \cos.\theta = 0, \quad \frac{dy}{dt} - V \sin.\theta = 0,$$

$$\frac{d^2x}{dt^2} = P \cos.a, \quad \frac{d^2y}{dt^2} = P \sin.a.$$

Hence, $\frac{dx}{dt}$ = velocity in $x$, $\frac{dy}{dt}$ = velocity in $y$,

$\frac{d^2x}{dt^2}$ = force in $x$, $\frac{d^2y}{dt^2}$ = force in $y$.

If we represent by $X$ and $Y$, the whole forces which act on the point in the direction of $x$ and of $y$, we have

$$\frac{d^2x}{dt^2} = X, \text{ and } \frac{d^2y}{dt^2} = Y;\ldots\ldots\ldots\ldots(a)$$

where $t$ is the independent variable; and $X$ and $Y$ are positive, when they tend to increase $x$ and $y$.

COR. 1. It is clear that if we had referred the path of the body to *three* rectangular co-ordinates, $x$, $y$, $z$, and if we had made $X$, $Y$, $Z$, represent the whole forces in the directions of these co-ordinates, we should have had, by reasoning exactly similar,

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y, \quad \frac{d^2z}{dt^2} = Z\ldots\ldots(a').$$

20. These equations enable us to solve various problems respecting the motions of bodies acted on by any forces. If the motion be known, we can, from them, find the forces in the directions of the co-ordinates, and by compounding these, the whole force which acts upon the body. If on the other hand, the force depends, in a known manner, on the position of the body, we can, by resolving it in the proper directions, find $X$, $Y$, $Z$, in terms of $x$, $y$, and $z$; and we shall then, by integrating the equations, have the motion of the body determined. If we can eliminate $t$, we

obtain a relation among the co-ordinates which defines the curve described by the body. We shall have instances of these various applications in what follows.

Ex. 1. To find the forces which must act upon a point, so that it may describe the arc of a parabola with a uniform motion.

If $x$, $y$, $s$, represent the abscissa, ordinate, and curve of the parabola, we shall have, since the velocity is constant,

$$\frac{ds}{dt} = c, \text{ a constant quantity;}$$

$$\therefore\ c^2 = \frac{ds^2}{dt^2} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}.$$

Now, if $4a$ be the principal parameter of the parabola, we have

$$y = 2\sqrt{ax}; \quad \therefore\ \frac{dy}{dx} = \sqrt{\frac{a}{x}};$$

Hence, $c^2 = \dfrac{dx^2}{dt^2}\left(1 + \dfrac{a}{x}\right)$, and $\dfrac{dx^2}{dt^2} = \dfrac{c^2}{1 + \dfrac{a}{x}}$;

Differentiating, $\dfrac{2dx}{dt} \cdot \dfrac{d^2x}{dt^2} = \dfrac{\dfrac{ac^2}{x^2} \cdot \dfrac{dx}{dt}}{\left(1 + \dfrac{a}{x}\right)^2}$;

$$\therefore\ \frac{d^2x}{dt^2} = \frac{c^2 a}{2(a+x)^2},$$

which gives the force parallel to the abscissa.

Again, $x = \dfrac{y^2}{4a}$; $\therefore\ \dfrac{dx}{dy} = \dfrac{y}{2a}$;

$$\therefore\ c^2 = \frac{dy^2}{dt^2}\left(\frac{y^2}{4a^2} + 1\right);$$

$$\therefore\ \frac{dy^2}{dt^2} = \frac{4a^2c^2}{4a^2 + y^2};$$

and differentiating, $\dfrac{2dy}{dt}\cdot\dfrac{d^2y}{dt^2} = -\dfrac{8a^2c^2y\dfrac{dy}{dt}}{(4a^2+y^2)^2}$;

$$\therefore \frac{d^2y}{dt^2} = -\frac{4a^2c^2y}{(4a^2+y^2)^2} = -\frac{4a^2c^2y}{(4a^2+4ax)^2} = -\frac{c^2y}{4(a+x)^2};$$

which gives the force parallel to the ordinate, the negative sign shewing that it tends towards the axis.

If $S$, fig. 8, be the focus, $A$ the vertex, and $SA$ the axis of the parabola; $AD = AS = a$, and $DN$ perpendicular to $AD$, so that $DN$ is the directrix; we shall have $DM$ or $NP = a + x$. Hence, the forces in $NP$ and $PM$ are respectively as

$$\frac{2AS}{PN^2}, \text{ and } \frac{PM}{PN^2}; \text{ or as } \frac{MK}{PN^2}, \text{ and } \frac{PM}{PN^2};$$

$PK$ being the normal, and therefore $MK = 2AS$. Hence, the whole force on $P$, which is compounded of these two, is in the direction $PK$*, and proportional to $\dfrac{PK}{PN^2}$.

Ex. 2. A body is acted upon at every point of its path, by a force which is proportional to the distance from a given centre towards which it tends: to find its path.

Let the centre of force $C$, fig. 9, be made the origin of rectangular co-ordinates $CM$, $MP$: the force in the direction $PC$ is every where proportional to $PC$. Resolve it in the directions $PM$, $MC$; and these lines will be proportional to the resolved parts, Hence, we shall have

force in direction of $x = -\mu x$, force in direction of $y = -\mu y$:

$\mu$ being some constant quantity, and the negative signs indicating the direction of the forces.

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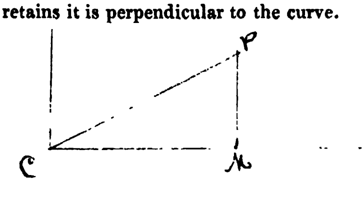
* It may easily be shewn that if a body move uniformly in *any* curve, the force which retains it is perpendicular to the curve.



Hence, $\dfrac{d^2x}{dt^2} = -mx$, $\dfrac{d^2y}{dt^2} = -my$;

$$\therefore 2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = -2mx\cdot\frac{dx}{dt}, \quad 2\frac{dy}{dt}\cdot\frac{d^2y}{dt^2} = -2my\cdot\frac{dy}{dt};$$

integrating, $\dfrac{dx^2}{dt^2} = C - mx^2$; $\dfrac{dy^2}{dt^2} = D - my^2$:

where $C$, $D$ are arbitrary quantities depending on the velocity and direction of the body's motion at some given point. We may evidently, without restricting their values, put for $C$ and $D$, $mh^2$ and $mk^2$; and thus we have

$$\frac{dx}{dt} = \sqrt{(C - mx^2)}, \quad \frac{dy}{dt} = \sqrt{(D - my^2)};$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sqrt{(D - my^2)}}{\sqrt{(C - mx^2)}} = \frac{\sqrt{(k^2 - y^2)}}{\sqrt{(h^2 - x^2)}}$$

$$\text{or } \frac{1}{\sqrt{(h^2 - x^2)}} = \frac{\frac{dy}{dx}}{\sqrt{(k^2 - y^2)}}.$$

Integrating with regard to $x$, we have $\alpha = \beta + \gamma$; where $\alpha$ is the arc whose sine is $\dfrac{x}{h}$, $\beta$ is the arc whose sine is $\dfrac{y}{k}$, and $\gamma$ an arbitrary arc. Therefore, we have

$$\sin.\,\alpha = \sin.\,\beta \cos.\,\gamma + \sin.\,\gamma \cos.\,\beta:$$

or if $n$ be the cosine of $\gamma$, and consequently $\sqrt{(1 - n^2)}$ its sine,

$$\frac{x}{h} = \frac{ny}{k} + \sqrt{(1 - n^2)}\cdot\sqrt{\left(1 - \frac{y^2}{k^2}\right)}:$$

transposing and squaring, we get

$$\frac{x^2}{h^2} + \frac{n^2y^2}{k^2} - \frac{2nxy}{hk} = (1 - n^2)\cdot\left(1 - \frac{y^2}{k^2}\right) = 1 - n^2 - \frac{y^2}{k^2} + \frac{n^2y^2}{k^2}.$$

Whence $\frac{x^2}{h^2}+\frac{y^2}{k^2}-\frac{2nxy}{hk}=1-n^2$:

which is the equation to an ellipse referred to rectangular co-ordinates measured from the centre. (*Wood's* Alg. Part IV.) Hence, the curve described by the body is an ellipse, of which the centre is $C$.

The axes of the ellipse may be thus found: let the tangent at $P$, fig. 9, be parallel to $Cx$, whence $CP$ and $CD$ will be conjugate diameters, and hence, by Conics,

$$CP^2+CD^2=a^2+b^2,\quad PM\,.\,CD=a\,.\,b,$$

where $a$ and $b$ are the semi-axes.

Now, since $\frac{dy}{dx}=\sqrt{\frac{k^2-y^2}{h^2-x^2}}$;

it is manifest that when the tangent is parallel to $Cx$, we have $y=k$: hence to find $x=CM$, we have, putting $k$ for $y$,

$$\frac{x^2}{h^2}+1-\frac{2nx}{h}=1-n^2;$$

$$\therefore\ \frac{x}{h}-n=0,\ x=nh=CM;\ \therefore\ CP^2=CM^2+MP^2=n^2h^2+k^2.$$

Also, to find $CD$, put $y=0$, and we have

$$\frac{x^2}{h^2}=1-n^2;\quad \therefore\ x=h\sqrt{(1-n^2)}=CD.$$

Hence $a^2+b^2=h^2+k^2$, $ab=kh\sqrt{(1-n^2)}$, whence $a$, $b$ are known.

To find the position of the major axis $CA$, or the angle $ACx=\theta$, we may proceed thus. Differentiating the equation to the curve, we find

$$\frac{dy}{dx}=-\frac{\frac{x}{h^2}-\frac{ny}{hk}}{\frac{y}{k^2}-\frac{nx}{hk}}.$$

Now at the point $A$, the curve is perpendicular to $CA$; and hence, at that point the normal passes through the centre; therefore (*Lacroix*, Art. 65.)

$$y\frac{dy}{dx} = -x; \quad \therefore \frac{\frac{x}{h^2} - \frac{ny}{hk}}{\frac{y}{k^2} - \frac{nx}{hk}} = \frac{x}{y};$$

$$\therefore \frac{xy}{h^2} - \frac{ny^2}{hk} = \frac{xy}{k^2} - \frac{nx^2}{hk};$$

whence we find $\frac{y^2}{x^2} + \frac{h^2 - k^2}{nhk}\,\frac{y}{x} - 1 = 0;$

and hence $\frac{2nhk}{h^2 - k^2} = \frac{2\frac{y}{x}}{1 - \frac{y^2}{x^2}} = \frac{2\tan.\theta}{1 - \tan.^2\theta} = \tan.2\theta;$

hence $\theta$ is known.

To find the time of describing any portion, we have

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} = \frac{1}{\sqrt{(C - mx^2)}} = \frac{1}{\sqrt{m}\sqrt{(h^2 - x^2)}};$$

$$\therefore t = \frac{1}{\sqrt{m}}.\sin.^{-1}\frac{x}{h} + \text{const.}$$

the constant quantity being determined by the place of the body at a given time.

For a whole revolution, we have time $= \frac{1}{\sqrt{m}}.2\pi$. Hence, the time of a revolution is independent in the size of the orbit.

# CHAP. III.

## CENTRAL FORCES.

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21. THE equations of the preceding Chapter would enable us to determine the motions of bodies acted on by any forces whatever, and of course, among the rest, by a force which is supposed always to tend to a centre, and to be represented by some function of the distance from that point; of which we had an instance in the last example. But problems respecting the action of *central forces*, as these are called, are of great importance, and lead to considerable simplifications of our general formula; we shall therefore treat separately this application of our reasonings.

### *Sect. I.* GENERAL THEOREMS.

22. PROP. *A body acted on by a central force will describe a curve lying in one plane, and will describe about the centre of force areas proportional to the times.*

Let the centre of force $S$ be made the origin of rectangular co-ordinates; and let $P$ be the force at a point $P$, of which the co-ordinates are $x$, $y$, $z$. If $r$ be the line joining the origin of co-ordinates $S$ with the point $P$, $x$, $y$, $z$ will be the edges of a parallelopiped, of which $r$ is the diagonal. Hence by Mechanics, Art. 31, $r : x :: P$ : resolved part of $P$ in the direction of the line $x$; and therefore this resolved part is $\frac{Px}{r}$. In like manner $\frac{Py}{r}$ and $\frac{Pz}{r}$ are the resolved parts of the force in the direction of $y$ and $z$. And these quantities must be made negative in order to represent the forces, because the forces tend to diminish $x$, $y$ and $z$. Hence we have by equations ($a'$), Art. 19,

$$\frac{d^2x}{dt^2} = -\frac{Px}{r}, \quad \frac{d^2y}{dt^2} = -\frac{Py}{r}, \quad \frac{d^2z}{dt^2} = -\frac{Pz}{r}.$$

Multiply the first by $y$ and the second by $x$, and subtract; we have thus

$$y\frac{d^2x}{dt^2} - x\frac{d^2y}{dt^2} = 0;$$

in like manner, $$z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2} = 0,$$

$$y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2} = 0.$$

But if we differentiate $y\frac{dx}{dt} - x\frac{dy}{dt}$ with regard to $t$, we have the differential

$$= y\frac{d^2x}{dt^2} + \frac{dy}{dt}\cdot\frac{dx}{dt} - \frac{dx}{dt}\cdot\frac{dy}{dt} - x\frac{d^2y}{dt^2} = y\frac{d^2x}{dt^2} - x\frac{d^2y}{dt^2}.$$

Hence it appears that the integral, with regard to $t$, of the left-hand side of the first of the above three equations, is

$$y\frac{dx}{dt} - x\frac{dy}{dt}.$$

The integral of the right-hand side is a constant quantity, which we may call $h$. Hence we have

$$y\frac{dx}{dt} - x\frac{dy}{dt} = h.$$

In like manner, $$z\frac{dx}{dt} - x\frac{dz}{dt} = h',$$

$$y\frac{dz}{dt} - z\frac{dy}{dt} = h'' \;\ldots\ldots\ldots\ldots\ldots\ldots (b')$$

where $h'$ and $h''$ are also constant quantities.

If we multiply the first of these equations by $z$, the second by $y$, the third by $x$, and add them, we find

$$0 = hz + h'y + h''x.$$

This is the equation to a plane passing through the origin. It appears, therefore, that the motion of the body takes place in a fixed plane passing through the centre of force.

The motion of the body will be the same, whatever be the position of the co-ordinate planes to which we refer it. Let the fixed plane in which it moves be taken for the plane of $xy$; then $z$ is everywhere $= 0$; and the only one of the preceding three equations now applicable is this,

$$y\frac{dx}{dt} - x\frac{dy}{dt} = h \ldots\ldots\ldots\ldots\ldots\ldots (b).$$

But if, in fig. 10, $SM = x$, $MP = y$, we have $-y\dfrac{dx}{dt}$ for the differential coefficient of the area $AMP$ with regard to $t$. Also, triangle $SMP = \frac{1}{2}xy$, and its differential coefficient

$$= \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right).$$

But sector $ASP = SMP + AMP$, and therefore differential coefficient of sector $ASP =$ differential coefficient of $SMP$ + differential coefficient of $AMP$

$$= \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right) - y\frac{dx}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

$$= \frac{h}{2}.$$

Hence the differential coefficient of $ASP$ taken with regard to the time is constant.

Therefore the velocity with which $ASP$ increases is constant, and the area $ASP$ is proportional to the time of the motion from $A$ to $P$.

Or, sector $ASP = \dfrac{ht}{2}$; $t$ being 0 at $A$:

and any portion of this sector is proportional to the time of describing that portion.

COR. 1. We have $ht = 2$ area $ASP$ described in $t$. Hence, if we make $t = 1$, $h = 2$ area described in time 1.

COR. 2. If $SP = r$, and angle $ASP = v$, we have (*Lacroix*,) differential of sector $ASP$ with regard to $v = \frac{1}{2}r^2$.

$$\text{Now,} \quad \frac{d.ASP}{dt} = \frac{d.ASP}{dv}.\frac{dv}{dt},$$

$$\text{or } \tfrac{1}{2}h = \tfrac{1}{2}r^2\frac{dv}{dt}, \quad \text{and } \frac{dv}{dt} = \frac{h}{r^2}.$$

COR. 3. When $P$ is referred to three co-ordinates, $z$ is perpendicular on the plane $xy$, and $x$, $y$ are the co-ordinates of the point where it falls. This point will therefore be the *projection* of the place of the body on the plane $xy$: and the area, of which the co-ordinates are $x$, $y$, will be the projected path. Now, in this case, as in last page,

$$\tfrac{1}{2}\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right),$$

is the differential coefficient of the sector described, taken with regard to the time.

And by the equation $y\dfrac{dx}{dt} - x\dfrac{dy}{dt} = h$, this differential coefficient is constant, and $= \frac{1}{2}h$. Therefore the sectorial area of the path of the body projected on the plane of $xy$ corresponding to $t$, will be $\frac{1}{2}ht$.

In like manner, $\frac{1}{2}\left(z\dfrac{dx}{dt} - x\dfrac{dz}{dt}\right) = \dfrac{h'}{2}$ is the differential coefficient of the area of the path on the plane $x$, and the area corresponding to $t$ is $\frac{1}{2}h't$.

And the area of the projection on the plane of $yz$ is in like manner $\frac{1}{2}h''t$.

23. PROP. *A body being acted on by a central force, it is required to find the velocity at any point.*

Let the motion be in the plane $x$, $y$, and let the centre of force be at the origin. Then we have

$$\frac{d^2x}{dt^2} = -\frac{Px}{r}, \quad \frac{d^2y}{dt^2} = -\frac{Py}{r}.$$

Multiply these by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$, respectively, and add; therefore

$$2\left(\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} + \frac{dy}{dt}\cdot\frac{d^2y}{dt^2}\right) = -2P\left(\frac{x}{r}\cdot\frac{dx}{dt} + \frac{y}{r}\cdot\frac{dy}{dt}\right).$$

But the left-hand side is the differential coefficient, with regard to $t$, of $\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}$. And since $x^2 + y^2 = r^2$,

$$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}:$$

hence the right-hand side $= -2P\frac{dr}{dt}$. Therefore,

$$-2P\frac{dr}{dt} = \frac{d\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right)}{dt};$$

$$= \frac{d\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right)}{dr}\cdot\frac{dr}{dt};$$

hence, dividing by $\frac{dr}{dt}$,

$$-2P = \frac{d\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right)}{dr}; \text{ and integrating}$$

$$C - 2\int_r P = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}.$$

If $s$ be the length of the arc, velocity $= \frac{ds}{dt}$;

$$\text{also, } \frac{ds^2}{dt^2} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2};$$

$$\text{hence, } \frac{ds^2}{dt^2} = C - 2\int_r P.$$

When $P$ is a function of $r$, the integration may be performed.

If the velocity for a given value of $r$ be known, $C$ may be found.

Cor. 1. If the velocity at a given distance $a$ from the centre be given, ($= A$ suppose), we have, at the distance $r$,

$$\frac{ds^2}{dt^2} = A^2 - 2\int_{r,a} P,$$

when $\int_{r,a}$ indicates that the integral begins when $r = a$.

Hence the velocity at the distance $r$ depends only on the distance, and is independent of the curve described: for $\int_{r,a} P$ does not depend on the curve.

Cor. 2. If we had retained the three co-ordinates $x$, $y$, $z$, and proceeded in the same manner as in this proposition, we should have had $r^2 = x^2 + y^2 + z^2$; and should have found

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = C - 2\int_r P \ldots\ldots (c').$$

Cor. 3. If the angle $ASP = v$.

$$\frac{ds^2}{dt^2} = \frac{dr^2}{dt^2} + r^2\frac{dv^2}{dt^2};$$

$$\text{hence, } \frac{d\left(\frac{dr^2}{dt^2} + r^2\frac{dv^2}{dt^2}\right)}{dr} = -2P.$$

24. Prop. *A body being acted upon by a central force, it is required to find the polar equation to the curve.*

By Cor. 2, to Art. 22, $\frac{dv}{dt} = \frac{h}{r^2}$;

$$\text{therefore, } \frac{dr^2}{dt^2} + r^2 \frac{dv^2}{dt^2} = \frac{dr^2}{dv^2} \cdot \frac{dv^2}{dt^2} + r^2 \frac{dv^2}{dt^2}$$

$$= \frac{h^2}{r^4} \frac{dr^2}{dv^2} + \frac{h^2}{r^2}.$$

Hence, by Cor. 3, to Art. 23,

$$\frac{d\left(\frac{h^2}{r^4} \frac{dr^2}{dv^2} + \frac{h^2}{r^2}\right)}{dr} = -2P.$$

Make $\frac{1}{r} = u$, whence $\frac{1}{r^2} \frac{dr}{dv} = -\frac{du}{dv}$; and the equation becomes

$$h^2 \frac{d\left(\frac{du^2}{dv^2} + u^2\right)}{dr} = -2P,$$

$$\text{or } h^2 \frac{d\left(\frac{du^2}{dv^2} + u^2\right)}{dv} \cdot \frac{dv}{dr} = -2P$$

$$= -2P \cdot \frac{dv}{dr} \cdot \frac{dr}{dv}.$$

Dividing out $h^2 \frac{dv}{dr}$, and observing that

$$\frac{dr}{dv} = -r^2 \frac{du}{dv} = -\frac{1}{u^2} \frac{du}{dv}, \text{ we find}$$

$$\frac{d\left(\frac{du^2}{dv^2} + u^2\right)}{dv} = \frac{2P}{h^2 u^2} \frac{du}{dv}.$$

$$\text{Differentiating, } 2 \frac{du}{dv} \cdot \frac{d^2u}{dv^2} + 2u \frac{du}{dv} = \frac{2P}{h^2 u^2} \cdot \frac{du}{dv};$$

$$\text{or dividing and transposing, } \frac{d^2u}{dv^2} + u - \frac{P}{h^2 u^2} = 0 \ldots\ldots\ldots (d),$$

$v$ being now the independent variable.

This is the equation of which we shall make use most commonly in the consideration of orbits described about a centre. It may be employed either in determining the law of force, when we know the curve, and consequently the relation of $u$ and $v$, which is called the direct problem of central forces; or if $P$ be known as a function of $r$, and therefore of $u$, we may, by integrating, find the relation of $u$ and $v$, which gives us the nature of the orbit; this is called the inverse problem of central forces.

25. PROP. *The orbit being given, to find the time of describing any part of it.*

By Cor. 2, Art. 22, $\dfrac{dv}{dt} = \dfrac{h}{r^2}$;

$$\text{hence, } \frac{dt}{dv} = \frac{r^2}{h};$$

and if $r$ be expressed in terms of $v$, $t = \int_v \dfrac{r^2}{h}$.

Also, $\dfrac{dt}{dr} = \dfrac{dt}{dv} \cdot \dfrac{dv}{dr}$, whence $\dfrac{dt}{dr} = \dfrac{r^2}{h} \cdot \dfrac{dv}{dr}$.

And if $v$ be expressed in terms of $r$, $t = \int_r \dfrac{r^2}{h} \cdot \dfrac{dv}{dr}$.

COR. 1. To find the velocity at any point in terms of $u$ and $v$,

$$\frac{ds^2}{dt^2} = \frac{dr^2}{dt^2} + r^2 \frac{dv^2}{dt^2} = \frac{dr^2}{dv^2} \cdot \frac{dv^2}{dt^2} + r^2 \frac{dv^2}{dt^2},$$

$$\left(\text{since } \frac{h}{r^2} = \frac{dv}{dt}\right), \qquad = \frac{h^2}{r^4} \cdot \frac{dr^2}{dv^2} + \frac{h^2}{r^2}$$

$$\text{or, vel.}^2 = h^2 \left(\frac{du^2}{dv^2} + u^2\right),$$

by Art. 24.

COR. 2. If we draw $SY$ a perpendicular on the tangent, and suppose $SY = p$, it is easily seen that we have

$$\frac{1}{r}\frac{ds}{dv} = \frac{r}{p}, \text{ or } \frac{ds}{dt}\cdot\frac{dt}{dv} = \frac{r^2}{p};$$

$$\text{whence } \frac{ds}{dt} = \frac{r^2}{p}\cdot\frac{dv}{dt} = \frac{r^2}{p}\cdot\frac{h}{r^2}, \text{ by Cor. 2, Art. 22;}$$

$$\text{hence } \frac{ds}{dt} = \frac{h}{p}.$$

Therefore *the velocity is inversely as the perpendicular on the tangent.*

26. It has been usual among English Mathematicians to define a spiral by the equation between the radius vector $r$, and the perpendicular on the tangent $p$. This is virtually only a differential equation to the curve, but its use is sometimes convenient.

Prop. *To obtain the central force in terms of* r *and* p.

By Cor. 2, Art. 25, we have $\frac{h^2}{p^2} = \frac{ds^2}{dt^2}$;

and differentiating with regard to $r$,

$$-\frac{2h^2}{p^3}\cdot\frac{dp}{dr} = \frac{d\frac{ds^2}{dt^2}}{dr} = -2P, \text{ by Art. 23;}$$

$$\therefore P = \frac{h^2}{p^3}\cdot\frac{dp}{dr}.$$

Cor. The velocity in the curve at any point is equal to that generated by the force $P$ at that point, continued constant, and acting on a body to urge it through one-fourth the chord of curvature drawn at that point through the centre of force.

For by this Article,

$(\text{velocity})^2 = \frac{h^2}{p^2} = Pp\frac{dr}{dp} = 2P\cdot\frac{1}{4}$ chord of curvature, (for chord $= 2p\frac{dr}{dp}$. *Lacroix*, Note H.)

$=$ (velocity)$^2$ produced by force $P$ through $\frac{1}{4}$ chord, because for constant forces, $(\text{velocity})^2 = 2fs$.

### *Sect. II.* CIRCULAR MOTION.

27. PROP. *When bodies revolve in circles having the centre of force in the centre of the circle, to determine the periodic time.*

In this case $r$ is constant, and therefore $u$ constant, and hence

$$\frac{du}{dv} = 0, \quad \frac{d^2u}{dv^2} = 0,$$

and equation ($d$) Art. 24, becomes

$$u - \frac{P}{h^2u^2} = 0, \quad \text{or } P = h^2u^3 = \frac{h^2}{r^3}.$$

Also by Cor. 1, Art. 25, $(\text{velocity})^2 = h^2u^2 = \frac{h^2}{r^2}$; hence $(\text{velocity})^2 = Pr$, and velocity $= \sqrt{Pr}$.

Now if $T$ be the time of a revolution

$$T = \frac{\text{circumference}}{\text{velocity}} = \frac{2\pi r}{\sqrt{Pr}} = \frac{2\pi\sqrt{r}}{\sqrt{P}}.$$

COR. 1. If $P \propto r$, $T$ is constant, velocity $\propto r$,

$P \propto 1$, $T \propto \sqrt{r}$, velocity $\propto \sqrt{r}$,

$P \propto \frac{1}{r^2}$, $T \propto r^{\frac{3}{2}}$, velocity $\propto \frac{1}{\sqrt{r}}$,

$P \propto \frac{1}{r^3}$, $T \propto r^2$, velocity $\propto \frac{1}{r}$,

COR. 2. Similarly if the variation of $T$ were given, that of $P$ would be known.

### *Sect. III.* ELLIPTICAL MOTION.

28. PROP. *A body being acted upon by a central force which is inversely as the square of the distance; it is required to find the integrals of the general equations of motion.*

The centre of force being made the origin of co-ordinates, let $r$ be the distance of any point, and $\frac{\mu}{r^2}$ the force at that distance. Then $-\frac{\mu x}{r^3}$, $-\frac{\mu y}{r^3}$, $-\frac{\mu z}{r^3}$ are the forces which act upon the body; and the equations ($a'$) of Art. 12, give

$$0 = \frac{d^2x}{dt^2} + \frac{\mu x}{r^3}, \quad 0 = \frac{d^2y}{dt^2} + \frac{\mu y}{r^3}, \quad 0 = \frac{d^2z}{dt^2} + \frac{\mu z}{r^3}:$$

we have already found four integrals of these equations, namely, in Art. 22, the integrals ($b'$)

$$y\frac{dx}{dt} - x\frac{dy}{dt} = h, \quad z\frac{dx}{dt} - x\frac{dz}{dt} = h', \quad y\frac{dz}{dt} - z\frac{dy}{dt} = h'';$$

also, from Art. 23, Cor. 2, observing that here

$$C - 2\int_r P = C - 2\int_r \frac{\mu}{r^2} = \frac{\mu}{r} - \frac{\mu}{2a},$$

($2a$ being the value of $r$ for which the expression vanishes,)

we have the integral ($c'$) $\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = \frac{2\mu}{r} - \frac{\mu}{a}$.

In order to find other integrals, take the equations

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{r^3}, \qquad \frac{d^2y}{dt^2} = -\frac{\mu y}{r^3},$$

$$h' = z\frac{dx}{dt} - x\frac{dz}{dt}, \qquad h'' = y\frac{dz}{dt} - z\frac{dy}{dt},$$

multiply each member of the upper equations by the member which stands under it, and subtract the second equations from the first: we have thus

$$h'\frac{d^2x}{dt^2} - h''\frac{d^2y}{dt^2} = \frac{\mu x}{r^3}\left(x\frac{dz}{dt} - z\frac{dx}{dt}\right) - \frac{\mu y}{r^3}\left(z\frac{dy}{dt} - y\frac{dz}{dt}\right)$$

$$= \frac{\mu}{r^3}\left(r^2 . \frac{dz}{dt} - z\,\frac{x\,dx + y\,dy}{dt}\right)$$

$$= \frac{\mu}{r^2} . \left(r\frac{dz}{dt} - z\frac{dr}{dt}\right) = \mu . \frac{d\frac{z}{r}}{dt}.$$

Integrating, $h'\dfrac{dx}{dt} - h''\dfrac{dy}{dt} = \dfrac{\mu z}{r} + f$, $f$ being a constant.

By a similar process we find

$$h''\frac{dz}{dt} - h\frac{dx}{dt} = \frac{\mu y}{r} + f',$$

$$h\frac{dy}{dt} - h'\frac{dz}{dt} = \frac{\mu x}{r} + f''.$$

Hence, we have 7 integrals, and 7 constants:

$$(h,\ h',\ h'',\ f,\ f',\ f'',\ a)$$

but these are equivalent only to 5 independent equations and 5 constants. See *Mrs Somerville, Mec. Heav.* Art. 370.

The body will move in a conic section. But instead of proceeding with the determination of the motion founded on the preceding integrals, we shall obtain the solution by means of the equation ($d$) of Art. 24. The former method of finding this motion leads to the mode of investigation which is employed in the case of the Planetary Theory, the latter to that which is more commonly used in the Lunar Theory.

29. PROP. *Let the force be inversely as the square of the distance, or* $\mathrm{P} = \dfrac{\mu}{\mathrm{r}^2} = \mu\mathrm{u}^2$*: to determine the motion.*

Here, by ($d$), $\dfrac{d^2u}{dv^2} + u - \dfrac{\mu}{h^2} = 0.$

To integrate this, let $u - \dfrac{\mu}{h^2} = w$;

$$\therefore\ \frac{d^2w}{dv^2} + w = 0;$$

and if $\epsilon^{kv}$ represent a particular value of $w$, we have

$$k^2 + 1 = 0,\quad k = \pm\sqrt{-1}.$$

(*Lacroix*, Art. 280, 281.)

Hence, the general value is

$$w = C\epsilon^{v\sqrt{-1}} + C'\epsilon^{-v\sqrt{-1}}$$

$$= \frac{(C+C')}{2}\left(\epsilon^{v\sqrt{-1}} + \epsilon^{v\sqrt{-1}}\right) + \left(\frac{C-C'}{2}\right)\left(\epsilon^{v\sqrt{-1}} - \epsilon^{v\sqrt{-1}}\right)$$

$$\left(\text{making } C + C' = C_1, \quad C - C' = \frac{C_2}{\sqrt{-1}}\right),$$

$$= C_1 \cos. v + C_2 \sin. v.$$

Hence, $u = \frac{1}{r} = C_1 \cos. v + C_2 \sin. v + \frac{\mu}{h^2}$,

and $\frac{du}{dv} = - C_1 \sin. v + C_2 \cos. v.$

Now when $v = 0$, $\frac{du}{dv} = C_2$,

when $v = \pi$, $\frac{du}{dv} = - C_2$;

hence, between $v = 0$, and $v = \pi$, there must be a value of $v$ which makes $\frac{du}{dv} = 0$: let this value of $v$ be $a$;

$$\therefore \; - C_1 \sin. a + C_2 \cos. a = 0; \quad C_2 \cos. a = C_1 \sin. a;$$

$$\therefore \frac{1}{r} = \frac{C_1 \cos. a \cos. v + C_2 \cos. a \sin. v}{\cos. a} + \frac{\mu}{h^2}$$

$$= \frac{C_1}{\cos. a}(\cos. a \cos. v + \sin. a \sin. v) + \frac{\mu}{h^2}$$

$$= \frac{C_1 \cos. (v - a)}{\cos. a} + \frac{\mu}{h^2}.$$

Let $r'$ and $r''$ be the values of $r$, for $v = a$, and $v = \pi + a$; both being supposed positive; hence,

$$\frac{1}{r'} = \frac{C_1}{\cos. a} + \frac{\mu}{h^2},$$

$$\frac{1}{r''} = -\frac{C_1}{\cos. a} + \frac{\mu}{h^2};$$ and, adding and subtracting,

$$\frac{\mu}{h^2} = \frac{1}{2}\left\{\frac{1}{r'} + \frac{1}{r''}\right\}; \quad \frac{C_1}{\cos. a} = \frac{1}{2}\left\{\frac{1}{r'} + \frac{1}{r''}\right\}.$$

Hence, the equation becomes

$$\frac{1}{r} = \frac{1}{2}\left\{\frac{1}{r'} + \frac{1}{r''}\right\}\cos. (v - a) + \frac{1}{2}\left\{\frac{1}{r'} + \frac{1}{r''}\right\};$$

$$\therefore\ r = \frac{2\,r'\,r''}{r' + r'' + (r'' - r')\cos. (v - a)}.$$

Now $r'$, $r''$ are opposite parts of the same line: let $r'' + r' = 2\,a$, $r'' - r' = 2\,ae$; $\therefore\ r'r'' = a^2 - a^2e^2$, and

$$r = \frac{a\,(1 - e^2)}{1 + e\cos. (v - a)},$$

the equation to an ellipse, $v - a$ being measured from the vertex nearest to $S$*.

If $r''$ be less than $r'$, $e$ will be negative, and the angle $v - a$ will be measured from the larger portion of the axis.

The curve may assume different forms by the alteration of the arbitrary quantities $C_1$, $C_2$. If $\frac{1}{r}$ or $u$ ever become 0, or negative, the form of the curve is no longer an ellipse.

Now the two values corresponding to $v = a$, and $v = \pi + a$, being those for which $\frac{du}{dv} = 0$, are manifestly the greatest and least values of $u$. Hence, if any value of $u$ be negative, one of these will be so. And hence the curve will no longer be an ellipse, if either

$$\frac{C_1}{\cos. a} + \frac{\mu}{h^2}, \text{ or } -\frac{C_1}{\cos. a} + \frac{\mu}{h^2}, \text{ be negative.}$$

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* *Principia*, Book I. Prop. 11.

If for instance, the latter be negative, we may suppose $\frac{1}{r''} = \frac{C_1}{\cos. a} - \frac{\mu}{h^2}$; where $r''$ is positive, whence, as before,

$$r = \frac{2r''r'}{(r''+r')\cos.(v-a)+r''-r'};$$

and making $r''-r'=2a$, $r''+r'=2ae$, (supposing $r''>r'$) we have $r = \frac{a(e^2-1)}{e\cos.(v-a)+1}$, the equation to an hyperbola.

If we have $-\frac{C_1}{\cos. a}+\frac{\mu}{h^2}=0$, we get, putting $\frac{\mu}{h^2}$ for $\frac{C_1}{\cos. a}$,

$\frac{1}{r} = \frac{\mu}{h^2}\{\cos.(v-a)+1\}$; and if $\frac{h^2}{\mu}=2D$,

$r = \frac{2D}{1+\cos.(v-a)}$; the equation to a parabola.

And similarly if $\frac{C_1}{\cos. a}+\frac{\mu}{h^2}=0$, the curve will be a parabola, but in a different position.

Hence, in all cases the curve will be a conic section: and supposing $C_1$ positive, it will be

an ellipse if $-\frac{C_1}{\cos. a}+\frac{\mu}{h^2}$ be positive, or $\frac{\mu}{h^2}>\frac{C_1}{\cos. a}$:

a parabola if $-\frac{C_1}{\cos. a}+\frac{\mu}{h^2}=0$, or $\frac{\mu}{h^2}=\frac{C_1}{\cos. a}$:

an hyperbola if $-\frac{C_1}{\cos. a}+\frac{\mu}{h^2}$ be negative, or $\frac{\mu}{h^2}<\frac{C_1}{\cos. a}$.

30. PROP. *A body being projected from a given point, with a given velocity, in a given direction; and acted on by a given force varying inversely as the square of the distance; to find the trajectory described**.

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* *Principia*, Book I. Prop. 17.

This problem might be solved by the preceding formulæ, but more simply as follows.

In fig. 11, let a body be projected from a distance $SP = R$, in a direction $PY$ making the angle $SPY = \delta$, and let the velocity at $P = V$; force $= \frac{\mu}{r^2}$, $Sp = r$.

By Art. 23, $\text{velocity}^2 = C - 2\int_r P = C - 2\int_r \frac{\mu}{r^2} = C + \frac{2\mu}{r}$;

and when $r = R$, velocity $= V$; $\therefore \text{velocity}^2 = V^2 + 2\mu\left\{\frac{1}{r} - \frac{1}{R}\right\}$.

But by Art. 25, Cor. 2, velocity $= \frac{h}{p}$, $p$ being the perpendicular on the tangent.

Hence, at $P$, $\frac{h^2}{R^2 \sin.^2 \delta} = V^2$; because perp$^r$. $= R \sin. \delta$

$$\text{and at } p,\ \frac{h^2}{p^2} = V^2 + 2\mu\left\{\frac{1}{r} - \frac{1}{R}\right\};$$

$$\therefore \text{dividing, } \frac{R^2 \sin.^2 \delta}{p^2} = 1 + \frac{2\mu}{V^2}\left\{\frac{1}{r} - \frac{1}{R}\right\};$$

$$\therefore \frac{1}{p^2} = \frac{1}{R^2 \sin.^2 \delta} - \frac{2\mu}{V^2 R^3 \sin.^2 \delta} + \frac{2\mu}{V^2 R^2 \sin.^2 \delta}\frac{1}{r};$$

now in the ellipse $\frac{1}{p^2} = -\frac{1}{b^2} + \frac{2a}{b^2} \cdot \frac{1}{r}$ *;

in the hyperbola $\frac{1}{p^2} = \frac{1}{b^2} + \frac{2a}{b^2} \cdot \frac{1}{r}$;

and the expression for $\frac{1}{p^2}$ in the trajectory may manifestly

* For $p^2 = \frac{b^2 r}{2a - r}$ in the ellipse,

and $p^2 = \frac{b^2 r}{2a + r}$ in the hyperbola, by Conic Sections.

be made to agree with one or the other of these, as the part of it independent of $r$ is positive or negative.

Hence, the trajectory will be an ellipse

$$\text{if } \frac{1}{R^2 \sin.^2 \delta} - \frac{2\mu}{V^2 R^3 \sin.^2 \delta} \text{ be negative;}$$

$$\text{that is, if } 1 < \frac{R}{V^2 R}; \quad \text{or if } V^2 < \frac{2\mu}{R}.$$

Similarly, if $V^2 > \frac{2\mu}{R}$, the curve will be an hyperbola.

If $V^2 = \frac{2\mu}{R}$, the curve will be a parabola.

In the case of the ellipse, we must have

$$\frac{1}{b^2} = \frac{2\mu}{V^2 R^3 \sin.^2 \delta} - \frac{1}{R^2 \sin.^2 \delta} = \frac{2\mu}{V^2 R^2 \sin.^2 \delta} \left\{ \frac{1}{R} - \frac{V^2}{2\mu} \right\};$$

$$\text{also } \frac{2a}{b^2} = \frac{2\mu}{V^2 R^2 \sin.^2 \delta};$$

$$\therefore \frac{1}{2a} = \frac{1}{R} - \frac{V^2}{2\mu}.$$

Hence, $a$ and $b$, the semi-axes of the ellipse, are known: and hence, $e^2 = \frac{a^2 - b^2}{a^2}$ is known. To find the position of the major axis we have

$$R = \frac{a(1 - e^2)}{1 + e \cos. (v - \alpha)};$$

where $v$ is the angle which determines the position of $P$, and is therefore known. Hence, $\cos. (v - \alpha)$ is known, and hence $\alpha$, which determines the position of the major axis.

Cor. 1. It appears by page 11, that the (velocity)$^2$ from an infinite distance $= \frac{2\mu}{R}$; hence, the trajectory will be an

ellipse, a parabola, or an hyperbola, as the velocity is less than, equal to, or greater than that acquired from infinity.

Cor. 2. In the first case

$$\frac{1}{2a} = \frac{1}{R} - \frac{V^2}{2\mu}; \quad \therefore \; V^2 = 2\mu\left\{\frac{1}{R} - \frac{1}{2a}\right\}:$$

therefore $V$ is the velocity acquired by falling from a distance $2a$ to a distance $D$, (see p. 11). Hence $2a$ is the distance from the centre at which a body must begin to fall, so that when it reaches the curve, it may have the velocity of the body in the curve; and this distance is the same for every point of the curve.

Cor. 3. Let the velocity $= n$ times the velocity from infinity, or

$$V^2 = n^2 . \frac{2\mu}{R}; \quad \therefore \text{ by last Cor. } \frac{1}{2a} = \frac{(1 - n^2)}{R};$$

$$2a = \frac{R}{1 - n^2}, \quad \frac{1}{b^2} = \frac{1 - n^2}{n^2 R^2 \sin.^2 \delta}, \quad b^2 = \frac{n^2 R^2 \sin.^2 \delta}{1 - n^2}.$$

Hence, $e^2 = 1 - \dfrac{b^2}{a^2} = 1 - 4n^2 (1 - n^2) \sin.^2 \delta.$

By means of these formulæ, we may, under given circumstances, find the magnitude and position of the trajectory.

Cor. 4. It appears from the preceding investigation that the major axis is independent of the direction of projection. And that, if $n$ be given, the eccentricity is independent of the distance of projection.

Ex. A body is projected in a direction making an angle of $30^0$ with the distance; and with a velocity which is to the velocity from infinity as 4 to 5: to determine the ellipse described.

In this case, $n = \dfrac{4}{5}$, $\quad \sin. \delta = \dfrac{1}{2}$;

$$\therefore\ 2a = \frac{R}{1-n^2} = \frac{25R}{9},$$

$$b^2 = \frac{n^2 R^2 \sin.^2 \delta}{1-n^2} = \frac{4R^2}{9};$$

$$\therefore\ \frac{b}{a} = \frac{12}{25} = .48\,; \quad \therefore\ e^2 = 1 - \frac{b^2}{a^2} = .7696$$

$$e = .877.$$

And at the point of projection

$$R = \frac{a(1-e^2)}{1+e\,.\cos.(v-\alpha)} = \frac{\frac{25R}{18} \times .2304}{1+.877\cos.(v-\alpha)};$$

$$\therefore\ \cos.(v-\alpha) = \frac{\frac{25}{18}\times .2304 - 1}{.877} = -\frac{680}{877}$$

$$= -\,.7753,\ \&c. = -\cos.39^0\,.\,10'.$$

Hence, $v - \alpha = 140^0\ 50' = ASP$, fig. 11.

31. Prop. *To find the time of describing any portion of the elliptical orbit**.

We have the equation $\dfrac{dt}{dv} = \dfrac{r^2}{h}$, (Art. 22); $\therefore\ t = \displaystyle\int_v \frac{r^2}{h}$;

and since by Art. 29, $\dfrac{\mu}{h^2} = \frac{1}{2}\left\{\dfrac{1}{r'} + \dfrac{1}{r''}\right\} = \frac{1}{2}\,.\,\dfrac{2a}{a^2 - a^2e^2}$;

$$\therefore\ h = \sqrt{(a\mu)}\,.\,\sqrt{(1-e^2)}.$$

Instead of substituting for $r$ its value in terms of $v$, which would produce for $\dfrac{dt}{dv}$ an expression not readily integrable; we shall express the time in terms of another angle $u$, as follows.

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* *Principia*, Book I. Prop. 31.

On the major axis of the ellipse $Aa$, fig. 13, let a semi-circle be described, and $MPQ$ drawn perpendicular to the axis, and $SQ$, $CQ$ joined: and let $ACQ = u$.

The expression $\int_v r^2$, beginning from $A$, is twice the area $ASP$. Now it is easily seen that

$$\frac{\text{area } ASP}{\text{area } ASQ} = \frac{MP}{MQ} = \frac{BC}{AC};$$

$$\therefore \text{ area } ASP = \frac{b}{a} . \text{ area } ASQ = \frac{b}{a}(\text{area } ACQ - SCQ)$$

$$= \frac{b}{a}(\tfrac{1}{2}AC . AQ - \tfrac{1}{2}SC . MQ)$$

$$= \frac{b}{2a}(a . au - ae . a \sin. u);$$

$$\therefore \int_v r^2 = 2 \text{ area } ASP = ab(u - e\sin. u),$$

and putting for $b$, $a\sqrt{(1-e^2)}$, and for $h$ its value,

$$t = \int_v \frac{r^2}{h} = \frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}(u - e\sin. u)\ldots\ldots\ldots\ldots(1),$$

the time being supposed to begin from $A$.

We have now to find the relation between $u$ and $v$;

$$HP^2 - HM^2 = PM^2 = SP^2 - SM^2;$$

$$\therefore HP^2 - SP^2 = HM^2 - SM^2;$$

$$(HP + SP)(HP - SP) = (HM + SM)(HM - SM);$$

$$2AC . (2AC - 2SP) = 2CM . 2CS;$$

or, dividing by $2 . 2$: and putting $a\cos. u$ for $CM$,

$$a(a - r) = a\cos. u . ae;$$

$$\therefore SP = r = a - ae\cos. u,$$

$$\text{and } \cos. v = \frac{SM}{SP} = \frac{CM - CS}{SP} = \frac{a\cos. u - ae}{a - ae\cos. u};$$

$$\therefore \cos.v = \frac{\cos.u - e}{1 - e\cos.u}.$$

Hence, $$\frac{1-\cos.v}{1+\cos.v} = \frac{1 - e\cos.u + e - \cos.u}{1 - e\cos.u - e + \cos.u}$$

$$= \frac{(1+e)(1-\cos.u)}{(1-e)(1+\cos.u)};$$

$$\therefore \tan.^2\frac{v}{2} = \frac{1+e}{1-e}.\tan.^2\frac{u}{2};$$

$$\text{and } \tan.\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}.\tan.\frac{u}{2}\ldots\ldots\ldots\ldots(2).$$

Hence, we can find $u$ in terms of $v$ by (2), and then $t$ in terms of $u$ by (1); and conversely. The angle $u$, or $ACQ$, is called in Astronomy the *eccentric anomaly*, $ASP$ being called the *true anomaly*.

COR. To find the time of a half revolution from $A$ to $a$, it is evident that we must take $u$ from 0 to $\pi$: which will give

$$t = \frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}.\pi.$$

Hence, the time of a revolution is $\dfrac{2a^{\frac{3}{2}}\pi}{\mu^{\frac{1}{2}}}$.

If the trajectory be an hyperbola, the calculations will be nearly the same as in the case of the ellipse.

If the force be repulsive, an hyperbola will be described, having the centre of force in its exterior focus; and its properties will be analogous to those in the other cases.

32. PROP. *To expand* u, *sin.* u, *sin.*2u, *in terms of* t.

By the last Article, we have

$$u - e\sin.u = \frac{\mu^{\frac{1}{2}}}{a^{\frac{3}{2}}}t = nt, \quad \text{if } n = \frac{\mu^{\frac{1}{2}}}{a^{\frac{3}{2}}}.$$

Hence, $u = nt + e\sin.u$.

But by *Lagrange's* Theorem, (Trans. *Lacroix*, Note Z, p. 635).

If $y = x + e\phi(y)$, we have [putting $y$ for $f(y)$],

$$y = x + \frac{e}{1}\phi(x) + \frac{e^2}{1.2}\frac{d\{\phi(x)\}^2}{dx} + \frac{e^2}{1.2.3}\frac{d^2\{\phi(x)\}^3}{dx^2} + \&c.$$

Now, making $\phi(x) = \sin. x$, we have

$$\phi(x) = \sin. x,$$

$$\frac{d\{\phi(x)\}^2}{dx} = \frac{d \sin.^2 x}{dx} = 2 \sin. x \cos. x = 2 \sin. 2x,$$

$$\frac{d^2\{\phi(x)\}^3}{dx^2} = \frac{d^2 \sin.^3 x}{dx^2} = \frac{d\,(3 \sin.^2 x \cos. x)}{dx}$$

$$= 6 \sin. x \cos. x - 3 \sin.^3 x$$

$$= 6 \sin. x - 9 \sin.^3 x.$$

Now, $\sin. 3x = 3 \sin. x - 4 \sin.^3 x$, whence $\sin.^3 x = \dfrac{3 \sin. x - \sin. 3x}{4}$,

and $\dfrac{d^2 \sin.^3 x}{dx^2} = \dfrac{9 \sin. 3x - 3 \sin. x}{4} = \dfrac{3^2 \sin. 3x - 3 \sin. x}{2^2}$.

And, in like manner, the other terms may be reduced to sines of multiples of $x$. Hence, putting $nt$ for $x$, and $u$ for $y$,

$$u = nt + \frac{e}{1}\sin. nt + \frac{e^2}{1.2.2} 2 \sin. 2nt$$

$$+ \frac{e^3}{1.2.3.2^2}(3^2 \sin. 3nt - 3 \sin. nt) + \&c.$$

Again, by the same theorem, putting $\sin. y$ for $f(y)$,

$$\sin. y = \sin. x + \frac{e}{1}\phi(x)\frac{d \sin. x}{dx} + \frac{e^2}{1.2}\frac{d\{\phi(x)\}^2 \dfrac{d \sin. x}{dx}}{dx} + \&c.$$

hence $\dfrac{\epsilon^{v\sqrt{-1}}-1}{\epsilon^{v\sqrt{-1}}+1} = \sqrt{\dfrac{1+e}{1-e}} \cdot \dfrac{\epsilon^{u\sqrt{-1}}-1}{\epsilon^{u\sqrt{-1}}+1} = \beta\,\dfrac{x-1}{x+1}$.

if $\epsilon^{u\sqrt{-1}} = x$, $\sqrt{\dfrac{1+e}{1-e}} = \beta$;

$$e^{v\sqrt{-1}} = \frac{1+\beta\,\dfrac{x-1}{x+1}}{1-\beta\,\dfrac{x-1}{x+1}} = \frac{(\beta+1)x-(\beta-1)}{\beta+1-(\beta-1)x}$$

$$= x\,\frac{1-\dfrac{\beta-1}{\beta+1}\cdot\dfrac{1}{x}}{1-\dfrac{\beta-1}{\beta+1}\,x} = x\,\frac{1-\dfrac{\lambda}{x}}{1-\lambda x}, \quad \text{if } \frac{\beta-1}{\beta+1} = \lambda,$$

$$\text{hence, } v\sqrt{-1} = \mathrm{l}x + \mathrm{l}\left(1-\frac{\lambda}{x}\right) - \mathrm{l}(1-\lambda x)$$

$$= \mathrm{l}x - \frac{\lambda}{x} - \frac{\lambda^2}{2x^2} - \frac{\lambda^3}{3x^3} - \&\text{c.}$$

$$+ \lambda x + \frac{\lambda^2 x^2}{2} + \frac{\lambda^3 x^3}{5} + \&\text{c.}$$

But $\mathrm{l}\,x = u\sqrt{-1}$, $x - \dfrac{1}{x} = 2\sqrt{-1}\,\sin. u$,

$$x^2 - \frac{1}{x^2} = 2\sqrt{-1}\,\sin. 2u, \ \&\text{c.}$$

hence, $v = u + 2\lambda\,\sin. u + \dfrac{2\lambda^2}{2}\,\sin. 2u + \dfrac{2\lambda^3}{3}\,\sin. 3u + \&\text{c.}$

Here,

$$\lambda = \frac{\beta-1}{\beta+1} = \frac{\sqrt{\dfrac{1+e}{1-e}}-1}{\sqrt{\dfrac{1+e}{1-e}}+1} = \frac{\sqrt{1+e}-\sqrt{1-e}}{\sqrt{1+e}+\sqrt{1-e}} = \frac{e}{1+\sqrt{1-e^2}}.$$

And, since $\phi(x)$ is $\sin. x$,

$$\phi(x)\frac{d\sin. x}{dx} = \sin. x\cos. x = \tfrac{1}{2}\sin. 2x,$$

$$\frac{d.\{\phi(x)\}^2 \frac{d\sin. x}{dx}}{dx} = \frac{d.\sin.^2 x\cos. x}{dx} = \frac{3\sin. 3x - \sin. x}{2^2},$$

as above: and so on. Hence,

$$\sin. u = \sin. nt + \frac{e}{1.2}\sin. 2nt + \frac{e^2}{1.2.2^2}(3\sin. 3nt - \sin. nt) + \&c.$$

Again, by the same theorem, putting $\sin. 2y$ for $f(y)$,

$$\sin. 2y = \sin. 2x + \frac{e}{1}.\phi(x)\frac{d.\sin. 2x}{dx} + \&c.$$

$$= \sin. 2x + \frac{e}{1}\phi(x)\, 2\cos. 2x + \&c.$$

$$= \sin. 2x + \frac{e}{1}\sin. x . 2\cos. 2x + \&c.$$

And $\sin. x . 2\cos. 2x = 2\sin. x - 4\sin.^3 x = \sin. 3x - \sin. x$.

Hence we have

$$\sin. 2y = \sin. 2x + \frac{e}{1}(\sin. 3x - \sin. x) + \&c.$$

$$\text{or } \sin. 2u = \sin. 2nt + \frac{e}{1}(\sin. 3nt - \sin. nt) + \&c.$$

33. Prop. *To expand* v *in terms of* u.

We have the equation

$$\tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}.\tan\frac{u}{2}.$$

But (*Lacr.* 164,) $\sqrt{-1}.\tan x = \dfrac{\epsilon^{2x\sqrt{-1}} - 1}{\epsilon^{2x\sqrt{-1}} + 1}$;

hence $\dfrac{\epsilon^{v\sqrt{-1}}-1}{\epsilon^{v\sqrt{-1}}+1} = \sqrt{\dfrac{1+e}{1-e}} \cdot \dfrac{\epsilon^{u\sqrt{-1}}-1}{\epsilon^{u\sqrt{-1}}+1} = \beta\dfrac{x-1}{x+1}$.

$$\text{if } \epsilon^{u\sqrt{-1}} = x, \quad \sqrt{\frac{1+e}{1-e}} = \beta;$$

$$e^{v\sqrt{-1}} = \frac{1+\beta\dfrac{x-1}{x+1}}{1-\beta\dfrac{x-1}{x+1}} = \frac{(\beta+1)x-(\beta-1)}{\beta+1-(\beta-1)x}$$

$$= x\,\frac{1-\dfrac{\beta-1}{\beta+1}\cdot\dfrac{1}{x}}{1-\dfrac{\beta-1}{\beta+1}x} = x\,\frac{1-\dfrac{\lambda}{x}}{1-\lambda x}, \quad \text{if } \frac{\beta-1}{\beta+1} = \lambda,$$

$$\text{hence, } v\sqrt{-1} = \mathrm{l}\,x + \mathrm{l}\left(1-\frac{\lambda}{x}\right) - \mathrm{l}\,(1-\lambda x)$$

$$= \mathrm{l}\,x - \frac{\lambda}{x} - \frac{\lambda^2}{2x^2} - \frac{\lambda^3}{3x^3} - \&c.$$

$$+ \lambda x + \frac{\lambda^2 x^2}{2} + \frac{\lambda^3 x^3}{5} + \&c.$$

$$\text{But } \mathrm{l}\,x = u\sqrt{-1}, \quad x - \frac{1}{x} = 2\sqrt{-1}\ \sin. u,$$

$$x^2 - \frac{1}{x^2}. = 2\sqrt{-1}\ \sin. 2u, \ \&c.$$

$$\text{hence, } v = u + 2\lambda \sin. u + \frac{2\lambda^2}{2}\sin. 2u + \frac{2\lambda^3}{3}\sin. 3u + \&c.$$

Here,

$$\lambda = \frac{\beta-1}{\beta+1} = \frac{\sqrt{\dfrac{1+e}{1-e}}-1}{\sqrt{\dfrac{1+e}{1-e}}+1} = \frac{\sqrt{1+e}-\sqrt{1-e}}{\sqrt{1+e}+\sqrt{1-e}} = \frac{e}{1+\sqrt{1-e^2}}.$$

34. PROP. *To expand* v *in terms of* t.

Since, by Art. 33, $v = u + 2\lambda \sin. u + \frac{2\lambda^2}{2} \sin. 2u + \&c.$

Substituting for $u$, sin. $u$, sin. $2u$, &c. from Art. 32, we have

$$v = nt + \frac{e}{1} \sin. nt + \frac{e^2}{1.2.2} 2 \sin. 2nt$$
$$+ \frac{e^2}{1.2.3.2^2} (3^2 \sin. 3nt - 3 \sin. nt)$$
$$+ 2\lambda \left\{ \sin. nt + \frac{e}{1.2} \sin. 2nt \right.$$
$$\left. + \frac{e^2}{1.2.2^2} (3 \sin. 3nt - \sin. nt) \right\}$$
$$+ \frac{2\lambda^2}{2} \left\{ \sin. 2nt + \frac{e}{1} (\sin. 3nt - \sin. nt) \right\}$$
$$+ \&c.$$

Also $\lambda = \frac{e}{2} + \frac{e^3}{8} + \&c.$ by expanding.

Hence we have

$$v = nt + \left(2e - \frac{e^3}{4}\right) \sin. nt + \frac{5e^2}{4} \sin. 2nt + \frac{13e^3}{12} \sin. 3nt + \&c.$$

which is true as far as terms involving $e^3$.

35. PROP. *To expand* r *in terms of* t.

We have by Art. 31, $r = a(1 - e \cos. u)$.

Now, as before, in Art. 32, putting cos. $y$ for $f(y)$,

$$\cos. y = \cos. z + \frac{e}{1} \phi(z) \frac{d . \cos. z}{dz} + \frac{e^2}{1.2} \frac{d . \left\{\phi(z)\right\}^2 \frac{d \cos. z}{dz}}{dz}$$
$$+ \frac{e^3}{1.2.3} \frac{d^2 . \left\{\phi(z)\right\}^3 \frac{d \cos. z}{dz}}{dz} + \&c.$$

Making $\phi(z) = \sin. z$, we have

$$\phi(z)\,\frac{d\cos. z}{dz} = -\sin.^2 z = \frac{\cos. 2z - 1}{2},$$

$$\frac{d.\{\phi(z)\}^2\,\frac{d\cos. z}{dz}}{dz} = -\frac{d\sin.^3 z}{dz}$$

$$= -3\sin.^2 z\cos. z = -\frac{3}{2}\sin. z\sin. 2z$$

$$= \frac{3}{4}(\cos. 3z - \cos. z),$$

$$\frac{d^2.\{\phi(z)\}^3\,\frac{d\cos. z}{dz}}{dz} = -\frac{d^2\sin.^4 z}{dz} = -\frac{d.4\sin.^3 z\cos. z}{dz}$$

$$= -12\sin.^2 z\cos.^2 z + 4\sin.^4 e$$

$$= 4\sin.^2 z - 16\sin.^2 z\cos.^2 z = 4\sin.^2 z - 4\sin.^2 2z$$

$$= 2(1 - \cos. 2z) - 2(1 - \cos. 4z)$$

$$= 2\cos. 4z - 2\cos. 2z, \text{ \&c.}$$

Hence, putting $u$ for $y$ and $nt$ for $z$,

$$\cos. u = \cos. nt + \frac{e}{1}\,\frac{\cos. 2nt - 1}{2} + \frac{e^2}{1.2}.\frac{3}{4}(\cos. 3nt - \cos. nt)$$

$$+ \frac{e^2}{1.2.3}.2(\cos. 4nt - \cos. 2nt) + \text{\&c.}$$

Hence, $\frac{r}{a} = 1 - e\cos. u = 1 + \frac{e^2}{2} - e\cos. nt - \frac{e^2}{2}\cos. 2nt$

$$-\frac{e^2}{1.2}.\frac{3}{4}(\cos. 3nt - \cos. nt) + \text{\&c.}$$

36. Prop. *To express the chord of an elliptical arc in terms of the mean anomalies of its extremities.*

If $u$, $u'$, be the mean anomalies, $v$, $v'$ the true, $r$, $r'$ the radii vectors; by Art. 31,

$$\frac{r}{a} = 1 - e\cos. u, \ \cos. v = \frac{\cos. u - e}{1 - e\cos. u} = \frac{a}{r}(\cos. u - e),$$

$$\text{whence } \sin.^2 v = 1 - \cos.^2 v = \frac{(1 - e\cos. u)^2 - (\cos. u - e)^2}{(1 - e\cos. u)^2}$$

$$\frac{r^2 \sin.^2 v}{a^2} = 1 - e^2 - (1 - e^2)\cos.^2 u,$$

$$= (1 - e^2)(1 - \cos.^2 u) = (1 - e^2)\sin.^2 u,$$

$$\text{and } \sin. v = \frac{a}{r}\sqrt{(1 - e^2)}\,.\,\sin. u.$$

In like manner

$$\cos. v' = \frac{a}{r'}(\cos. u' - e), \ \sin. v' = \frac{a}{r'}\sqrt{(1 - e^2)}\sin. u'.$$

Now by trigonometry, if $c$ be the chord of the arc,

$$c^2 = (r'\cos. v' - r\cos. v)^2 + (r'\sin. v' - r\sin. v)^2$$

$$= a^2(\cos. u' - \cos. u)^2 + a^2(1 - e^2)(\sin. u' - \sin. u)^2,$$

by the preceding formulæ.

Let $\dfrac{u' - u}{2} = \beta$, $\dfrac{u' + u}{2} = \beta'$;

$$\text{then } u' = \beta' + \beta, \ u = \beta' - \beta;$$

$$\text{and } \cos. u' - \cos. u = 2\sin. \beta \sin. \beta',$$

$$\sin. u' - \sin. u = 2\sin. \beta\cos. \beta';$$

therefore

$$c^2 = 4a^2 \sin.^2 \beta \sin.^2 \beta' + 4a^2(1 - e^2)\sin.^2 \beta\cos.^2 \beta';$$

or, $c^2 = 4a^2 \sin.^2 \beta(1 - e^2\cos.^2 \beta')$.

37. PROP. *In an elliptical orbit, the time of describing any arc depends only on the chord of the arc, the sum of the extreme radii vectors, and the major axis: these quantities being the same, the time is the same, whatever be the eccentricity.*

If $v$ be the angle measured from the perihelion, we have, by Arts. 29, 31, 32, making $\mu = 1$,

$$r = \frac{a(1 - e^2)}{1 + e\cos. v},$$

$$r = a(1 - e\cos. u),$$

$$t = a^{\frac{3}{2}}(u - e\sin. u).$$

Suppose that $r$, $v$, $u$, $t$ belong to the first extremity of the elliptical arc, and that $r'$, $v'$, $u'$, $t'$ are the corresponding quantities for the other extremity. Then

$$r' = \frac{a(1 - e^2)}{1 + e\cos. v'},$$

$$r' = a(1 - e\cos. u'),$$

$$t' = a^{\frac{3}{2}}(u' - e\sin. u').$$

Hence, $$t' - t = a^{\frac{3}{2}}\{u' - u - e(\sin. u' - \sin. u)\}$$

$$= a^{\frac{3}{2}}\left\{u' - u - 2e\sin.\frac{u' - u}{2}\cos.\frac{u' + u}{2}\right\}.$$

Let $t' - t = \tau$, $\frac{u' - u}{2} = \beta$, $\frac{u' + u}{2} = \beta'$;

therefore, $\tau = 2a^{\frac{3}{2}}(\beta - e\sin.\beta\cos.\beta')$.........(1).

Add together the two second values of $r$ and $r'$, observing that $\cos. u' + \cos. u = 2\cos.\beta\cos.\beta'$, and making $r' + r = R$; we then have

$$R = 2a(1 - e\cos.\beta\cos.\beta')\text{.........(2).}$$

Now $c$ being the chord of the elliptical arc, we have by the last Prop.

$$c^2 = 4a^2 \sin.^2\beta (1 - e^2 \cos.^2\beta') \ldots\ldots\ldots (3).$$

From (2) we have $e \cos.\beta' = \dfrac{2a - R}{2a \cos.\beta}$.

This substituted in (1) and (3) gives

$$2\tau = 2a^{\frac{3}{2}} \left\{\beta + \frac{R - 2a}{2a} . \tan.\beta\right\} \ldots\ldots\ldots (4).$$

$$c^2 = 4a^2 \tan.^2\beta \left\{\cos.^2\beta - \left(\frac{2a - R}{2a}\right)^2\right\} \ldots\ldots (5).$$

Here $e$ is not involved: and if we take the value of $\beta$ given by (5) and substitute it in (4), we shall have $\tau$ depending on $a$, $c$, and $R$ solely.

38. Prop. *To express the time of describing any arc of an elliptical orbit in terms depending on the major axis, chord, and extreme radii vectors.*

Make $\quad x = \dfrac{2a - R + c}{2a}$, $x' = \dfrac{2a - R - c}{2a}$;

Hence, $xx' = \left(\dfrac{2a - R}{2a}\right)^2 - \dfrac{c^2}{4a^2}$

$$\frac{x^2 + x'^2}{2} = \left(\frac{2a - R}{2a}\right)^2 + \frac{c^2}{4a^2}.$$

For the sake of abbreviation, make $\dfrac{2a - R}{2a} = k$, $\sin.\beta = s$, $\tan.\beta = t$. Then by (5) of last Prop.

$$\frac{c^2}{4a^2} = s^2 - k^2 t^2.$$

Hence, $k^2(1+t^2)-s^2\left(=k^2-\frac{c^2}{4a^2}\right)=xx'$,

$$k^2(1-t^2)+s^2\left(=k^2+\frac{c^2}{4a^2}\right)=\frac{x^2+x'^2}{2}.$$

Square the first equation and subtract double the second, adding 1 to both sides: then

$$k^4(1+t^2)^2-2k^2(1+t^2)\,s^2+1+s^4$$
$$-2k^2(1-t^2)-2s^2=1-x^2-x'^2+x^2x'^2:\ \text{or,}$$
$$k^4(1+t^2)^2-2k^2(1+s^2-t^2+s^2t^2)+(1-s^2)^2=(1-x^2)(1-x'^2).$$

But $t^2=\frac{s^2}{1-s^2}$; whence, $s^2-t^2+s^2t^2=0$, and we may put $1-s^2+t^2-s^2t^2$ for $1+s^2-t^2+s^2t^2$. When this is done the first side becomes a square; and extracting the root we have

$$k^2(1+t^2)-(1-s^2)=\sqrt{(1-x^2)(1-x'^2)}.$$

Also, $k^2(1+t^2)-s^2=xx'$;

$$\text{therefore, } 1-2s^2=xx'\pm\sqrt{(1-x^2)(1-x'^2)}.$$

Let $\zeta=\cos.^{-1}x$, $\zeta'=\cos.^{-1}x'$;

$$\text{also, } 1-2s^2=\cos.2\beta;$$

$$\text{hence, } \sqrt{1-x^2}=\sin.\zeta,\quad \sqrt{1-x'^2}=\sin.\zeta';$$

and taking the upper sign,

$$\cos.2\beta=\cos.\zeta\cos.\zeta'+\sin.\zeta\sin.\zeta'$$
$$2\beta=\zeta'-\zeta;$$

$$\text{hence, } \tan.\beta=\frac{\sin.\frac{\zeta'-\zeta}{2}}{\cos.\frac{\zeta'-\zeta}{2}}=\frac{\sin.\frac{\zeta'-\zeta}{2}\cos.\frac{\zeta'+\zeta}{2}}{\cos.\frac{\zeta'-\zeta}{2}\cos.\frac{\zeta'+\zeta}{2}}$$

$$=\frac{\sin.\zeta'-\sin.\zeta}{\cos.\zeta'+\cos.\zeta}=\frac{\sin.\zeta'-\sin.\zeta}{x'+x}.$$

Also, $\varkappa + \varkappa' = \dfrac{2a - R}{a}$; hence by (4) of last Prop. $\tau$ becomes

$$\tau = a^{\frac{3}{2}} \{\zeta' - \zeta - \sin.\zeta' + \sin.\zeta\}.$$

In this case the whole period corresponds $a^{\frac{3}{2}} . 2\pi$ since $\mu$ is 1.

Hence, if $T$ be the period of a body for which $a = 1$;

$$\frac{\tau}{T} = a^{\frac{3}{2}} \frac{\zeta' - \zeta - \sin.\zeta' + \sin.\zeta}{2\pi}.$$

## *Sect. IV.* Parabolic Motion.

39. Prop. *In the case in which the orbit is a parabola, it is required to find the time in terms of the angle.*

As before, $\dfrac{dt}{dv} = \dfrac{r^2}{h}$;

and $r = \dfrac{2D}{1 + \cos.v}$, $v$ being measured from the vertex, so that $a$ in page 42, is 0.

Also, $h = \sqrt{2\mu D}$. Hence,

$$\frac{dt}{dv} = \frac{4D^2}{\sqrt{2\mu D}(1 + \cos.v)^2} = \frac{D^{\frac{3}{2}}}{(2\mu)^{\frac{1}{2}}} \cdot \frac{1}{\cos.^4 \frac{v}{2}}$$

$$= \frac{D^{\frac{3}{2}}}{(2\mu)^{\frac{1}{2}}} \frac{\cos.^2 \frac{v}{2} + \sin.^2 \frac{v}{2}}{\cos.^4 \frac{v}{2}} = \frac{D^{\frac{3}{2}}}{(2\mu)^{\frac{1}{2}}} \cdot \frac{1}{\cos.^2 \frac{v}{2}} \left\{1 + \tan.^2 \frac{v}{2}\right\},$$

$$\text{and } \frac{1}{\cos.^2 \frac{v}{2}} = \frac{2d \tan. \frac{v}{2}}{dv};$$

hence, integrating, $t = \dfrac{2D^{\frac{3}{2}}}{(2\mu)^{\frac{1}{2}}} \left\{\tan. \dfrac{v}{2} + \dfrac{1}{3} \tan.^3 \dfrac{v}{2}\right\}$,

$t$ being supposed to begin at the vertex, when $v = 0$.

40. Prop. *To find the place of a body in a given parabolic orbit, at a given time, by a geometrical construction.*

(Newton, Book I. Prop. xxx.)

Fig. 147. Let $S$ be the focus, $A$ the vertex, $t$ the time from $A$ when the body is at $P$. The area $ASP$

$$=\frac{1}{2}t.h=\frac{t}{2}\sqrt{2\mu D}.$$

Let $M$ be such that area $ASP = 4M.AS$.

$$\text{or } 4D.M=t\sqrt{\frac{\mu D}{2}};\quad M=t\frac{\mu^{\frac{1}{2}}D^{\frac{1}{2}}}{4\sqrt{2}}.$$

Bisect $AS$ in $G$, and make $GH$ perpendicular to $AS$ and equal to $3M$: with centre $H$, and radius $HA$, describe a circle $SP$, meeting the parabola in $P$: $P$ will be the place of the body at the time $t$.

Draw $PO$ perpendicular to the axis. Then

$$AG^2+GH^2=HP^2=(AO-AG)^2+(PO-GH)^2$$

$$=AO^2+PO^2-2GA.AO-2GH.PO+AG^2+GH^2.$$

Hence,

$$2GH.PO=AO^2+PO^2-2GA.AO=AO^2+PO^2-SA.AO$$

$$=AO^2+PO^2-\frac{1}{4}PO^2=AO^2+\frac{3}{4}PO^2$$

$$=AO.\frac{PO^2}{4AS}+\frac{3}{4}PO^2=\frac{3PO^2}{4AS}\left(\frac{AO}{3}+AS\right);$$

$$\text{or }\frac{4}{3}GH.AS=\frac{PO}{2}\left(\frac{AO}{3}+AS\right)=\frac{PO}{2}\left(\frac{AO}{3}+AO-SO\right)$$

$$=\frac{2}{3}AO.PO-\frac{1}{2}SO.PO=\text{area}AOP-\text{triangle}SOP,$$

because the area of a parabola is $\frac{2}{3}$ of the circumscribing rectangle.

Hence, $4M.AS$ = area $ASP$, and $P$ is rightly determined.

Cor. 1. Let $SQ$ be perpendicular to the axis, and let $M$ be $N$ when the body is at $Q$:

$$\text{then } 4N.AS = \text{area } ASQ = \frac{2}{3} AS.SQ = \frac{4}{3} AS^2.$$

$$\text{Hence, } N = \frac{AS}{3};$$

and time in $AQ$ : time in $AP$ :: $M$ : $N$ :: $3M$ : $3N$ :: $GH$ : $AS$.

Cor. 2. The line $GH$ increases proportionally to the time; hence the velocity of $H$ is uniform. Now, when the body is very near to $A$, $AP$ is small, area $ASP = \dfrac{AS.AP}{2}$ ultimately.

$$\text{And } GH = 3M = \frac{3 \text{ area } ASP}{4AS} = \frac{3AP}{8};$$

$$\text{hence, } \frac{GH}{AP} = \frac{3}{8} \text{ ultimately; and therefore } \frac{\text{vel. at } H}{\text{vel. at } A} = \frac{3}{8}.$$

41. Prop. *It is required to express the time of describing any arc of a parabola in terms of its chord, and of the rays at its extremities.* (Lambert's *Theorem.*)

The expression here sought is to be independent of the distance of the vertex from the focus.

Make in Art. 38. $a$ large:

$$\text{and since } x = 1 - \frac{R-c}{2a};$$

$$\sin.\zeta = \sqrt{1-x^2} = \sqrt{\left\{1 - \left(1 - \frac{R-c}{2a}\right)^2\right\}} = \sqrt{\frac{R-c}{a}}$$

ultimately when $a$ is very large:

$$\text{hence ultimately, } \zeta - \sin.\zeta = \frac{1}{6}\zeta^3 = \frac{1}{6}\left(\frac{R-c}{a}\right)^{\frac{3}{2}}.$$

$$\text{Similarly, } \zeta' - \sin.\zeta' = \frac{1}{6}\zeta'^3 = \frac{1}{6}\left(\frac{R+c}{a}\right)^{\frac{3}{2}}.$$

Hence, in this case, we have by Art. 38,

$$\frac{\tau}{T} = \frac{1}{12\pi}\left\{(R+c)^{\frac{3}{2}} \mp (R-c)^{\frac{3}{2}}\right\}$$

$$= \frac{1}{12\pi}\left\{(r+r'+c)^{\frac{3}{2}} \mp (r+r'-c)^{\frac{3}{2}}\right\}.$$

The sign + is to be taken when the two extremities of the chord $c$ are on different sides of the axis.

If we take the radius of the Earth's orbit for unity, and suppose the parabola to be described about the Sun, $T = 365{,}25$ days.

## *Sect. V.* The Inverse Cube.

42. Prop. *Let the force be inversely as the cube of the distance, or* $P = \frac{\mu}{r^3} = \mu u^3$*.

The equation in this case becomes

$$\frac{d^2u}{dv^2} + u - \frac{\mu u}{h^2} = 0.$$

To integrate this equation, let $u = \epsilon^{kv}$ be a particular solution. (See *Lacroix*, Elem. Treat. Art. 280.)

Thence, $k^2 + 1 - \frac{\mu}{h^2} = 0$;

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* Newton considered the curves described when the force is inversely as the cube of the distance, and besides the logarithmic spiral, noticed the curves, Species I, V, and VI; but omitted the examination of the others, by supposing the body to move from an apse. *Principia*, Book I. Prop. 9, and Prop. 41, Cor. 3. The complete analysis of this case was given by Cotes in his *Logometria*; Phil. Trans. 1715; from which circumstance these curves are sometimes called Cotes's Spirals. From certain analogies observed by Newton, Species I, and V, are called the Hyperbolic and Elliptic Spiral, respectively. It may be remarked, however, that the Reciprocal Spiral is sometimes, by foreign writers, called the Hyperbolic Spiral.

and if $\gamma$, $-\gamma$ be the two values of $k$ in this equation, the general integral will be

$$u = C\epsilon^{\gamma v} + C'\epsilon^{-\gamma v};$$

$C$, $C'$ being any two arbitrary constants.

The curves described will be different, as the values of $\pm\gamma$ are possible or impossible, and as the arbitrary constants are positive or negative. We shall consider the different species thus produced.

SPECIES I. Let $\frac{\mu}{h^2} > 1$, and $C$, $C'$ both the same sign.

Hence, $k = \pm\sqrt{\left(\frac{\mu}{h^2} - 1\right)} = \pm\gamma$.

Suppose, therefore, $u = C\epsilon^{\gamma v} + C'\epsilon^{-\gamma v}$;

$$\text{hence, } \frac{du}{dv} = \gamma\left\{C\epsilon^{\gamma v} - C'\epsilon^{-\gamma v}\right\}.$$

Now, when $\frac{du}{dv} = 0$, $C\epsilon^{\gamma v} = C'\epsilon^{-\gamma v}$, or $\epsilon^{2\gamma v} = \frac{C'}{C}$;
which can always be fulfilled by a possible value of $v$: let this value be $a$, so that

$$C\epsilon^{\gamma a} = C'\epsilon^{-\gamma a} = c; \quad \therefore \; C = c\epsilon^{-\gamma a}, \quad C' = c\epsilon^{\gamma a}.$$

Hence, $u = c\left\{\epsilon^{\gamma(v-a)} + \epsilon^{-\gamma(v-a)}\right\}$.

When $v = a$, $u = 2c$; and since at that point $\frac{du}{dv} = 0$, the curve is perpendicular to the radius, or there is an apse*. As $v$ increases, $u$ increases, and therefore $r$ diminishes, and when $v$ becomes infinite, $r$ becomes 0. Hence, the curve is such as is represented in fig. 14.

* An apse is a point where the curve is perpendicular to the radius vector, and where, consequently, in general, the radius vector will be either a maximum or minimum.

If $C$, $C'$ be both negative, the curve will be the same. The sign can only indicate that the angle $v$ is to be measured in the opposite direction.

SPECIES II. Let $\frac{\mu}{h^2} > 1$; and $C' = 0$.

Therefore, $u = C\epsilon^{\gamma v}$; $\gamma v = 1\frac{u}{C} = 1\frac{a}{r}$, if $a = \frac{1}{C}$.

Hence, the curve is the *logarithmic spiral*, fig. 15.

Differentiating, $\gamma\frac{dv}{dr} = -\frac{1}{r}$; $-\frac{1}{r}.\frac{dr}{dv} = \gamma = \sqrt{\left(\frac{\mu}{h^2} - 1\right)}$:

hence, $\sqrt{\left(\frac{\mu}{h^2} - 1\right)}$ is the co-tangent of the constant angle $SPY$, which the tangent makes with the radius vector: and therefore $\frac{\sqrt{\mu}}{h}$ is the co-secant, and $\frac{h}{\sqrt{\mu}}$ the sine of $SPY$.

$$\text{Let } \frac{h}{\sqrt{\mu}} = \sin.\beta; \quad \therefore\ h = \sin.\beta\sqrt{\mu}.$$

If $C = 0$, the curve will be the same.

SPECIES III. Let $\frac{\mu}{h^2} > 1$, and $C'$ negative.

Therefore, $u = C\epsilon^{\gamma v} - C'\epsilon^{-\gamma v}$.

Now, when $u = 0$, $C\epsilon^{\gamma v} = C'\epsilon^{-\gamma v}$ and $\epsilon^{2\gamma v} = \frac{C'}{C}$, for which there is always a possible value of $v$: let this value be $a$; and let $C\epsilon^{\gamma a} = C'\epsilon^{-\gamma a} = c$;

$$\therefore\ C = c\epsilon^{-\gamma a}; \quad C' = c\epsilon^{\gamma a}:$$

$$\text{and hence } u = c\left\{\epsilon^{\gamma(v-a)} - \epsilon^{-\gamma(v-a)}\right\}.$$

When $v = a$, since $u = 0$, $r$ is infinite. As $v$ increases, $u$ increases, and $r$ decreases; and when $v$ is infinite, $u$ is also infinite, and $r$ vanishes. Hence, the form of the curve is that in fig. 16, $v - a$ being the angle $ASP$.

If $p$ be the perpendicular from $S$ upon the tangent, we have, (p. 36,)

$$\frac{1}{p^2} = \frac{1}{r^4}\frac{ds^2}{dv^2} = \frac{1}{r^2} + \frac{1}{r^4}\frac{dr^2}{dv^2}$$

$$= c^2\left\{\epsilon^{\gamma(v-a)} - \epsilon^{-\gamma(v-a)}\right\}^2 + c^2\gamma^2\left\{\epsilon^{\gamma(v-a)} + \epsilon^{-\gamma(v-a)}\right\}^2$$

$$= c^2(1+\gamma^2)\left\{\epsilon^{2\gamma(v-a)} + \epsilon^{-2\gamma(v-a)}\right\} + 2c^2(\gamma^2 - 1).$$

and when $v = a$, $\frac{1}{p^2} = c^2(1+\gamma^2) \cdot 2 + 2c^2(\gamma^2 - 1)$

$$= 4c^2\gamma^2; \text{ and } p = \frac{1}{2c\gamma}.$$

And hence there is an asymptote $BZ$ to the curve, parallel to $SA$, at a distance $SB = \frac{1}{2c\gamma}$.

Similarly, if $C'$ be positive and $C$ negative.

Species IV. Let $\frac{\mu}{h^2} = 1$.

In this case we must return to the original equation, which here gives us

$$\frac{d^2u}{dv^2} = 0; \therefore \frac{du}{dv} = C, \ u = C(v-a): \ r = \frac{1}{C(v-a)} = \frac{a}{v-a};$$

it being supposed that when $u = 0$, $v = a$.

For this position $r$ is infinite: any other value of $r$, as $SP$, is reciprocally as the angle $v - a$, or $ASP$. Hence, the curve in this case is the *Reciprocal Spiral*, fig. 17.

If a circular arc $PQ$ be described with centre $S$, $PQ = r(v-a) = a$; and hence, is at every point the same.

It is manifest that the curve will have an asymptote $BZ$, such, that $SB = a$.

Species V. Let $\frac{m}{h^2} < 1$.

In this case the values of $k$ in the equation $k^2 + 1 - \frac{\mu}{h^2} = 0$, are impossible. Let them be $\pm\gamma\sqrt{-1}$. Therefore for the general integral of the equation we have

$$\begin{aligned} u &= C\epsilon^{\gamma v\sqrt{-1}} + C'\epsilon^{-\gamma v\sqrt{-1}} \\ &= \tfrac{1}{2}(C + C')\,(\epsilon^{\gamma v\sqrt{-1}} + \epsilon^{-\gamma v\sqrt{-1}}) \\ &\quad + \tfrac{1}{2}(C - C')\,(\epsilon^{\gamma v\sqrt{-1}} - \epsilon^{-\gamma v\sqrt{-1}}) \\ &= C_1 \cos.\gamma v + C_2 \sin.\gamma v, \end{aligned}$$

making $C_2 = C + C'$, and $C_2 = \sqrt{-1}\,(C - C')$.

$$\text{Hence, } \frac{du}{dv} = -\gamma C_1 \sin.\gamma v + \gamma C_2 \cos.\gamma v\,;$$

and when $\frac{du}{dv} = 0$, $\tan.\gamma v = \frac{C_2}{C_1}$: for which there is always a value of $v$, whether $C_1$ and $C_2$ be of the same or of different signs. Let $a$ be this value;

$$\therefore\ C_2 = C_1 \frac{\sin.\gamma a}{\cos.\gamma a}\,;$$

$$\begin{aligned} \therefore\ u &= \frac{C_1}{\cos.\gamma a}\{\cos.\gamma v \cos.\gamma a + \sin.\gamma v \sin.\gamma a\} \\ &= \frac{C_1}{\cos.\gamma a}\cos.\gamma(v - a), \end{aligned}$$

or, making $\frac{C_1}{\cos.\gamma a} = \frac{1}{a}$, $u = \frac{\cos.\gamma(v-a)}{a}$, and $r = \frac{a}{\cos.\gamma(v-a)}$.

When $v = a$, $r = a$, and there is an apse. When $\gamma(v - a) = \frac{\pi}{2}$, $r$ is infinite, and therefore may be parallel to an asymptote. To find the position of the asymptote, we have

$$\frac{1}{p^2} = u^2 + \frac{du^2}{dv^2} = \frac{\cos.^2\gamma(v - a)}{a^2} + \frac{\gamma^2 \sin.^2\gamma(v - a)}{a^2}\,;$$

and when $\gamma(v-a)=\frac{\pi}{2}$, $\frac{1}{p^2}=\frac{\gamma^2}{a^2}$, $p^2=\frac{a^2}{1-\frac{\mu}{h^2}}=SB^2$.

The form of the curve is given in fig. 18.

43. Prop. *To determine in what cases each of these curves will be described.*

We may observe, that in the case where the body describes a circle, and consequently where $\frac{d^2u}{dv^2}=0$, we have,

$$u-\frac{\mu u}{h^2}=0, \text{ and } \frac{\mu}{h^2}=1, \text{ or } h=\sqrt{\mu}, \text{ the area in time } 1.$$

Now the species varies as $\frac{\mu}{h^2}$ in the curve is greater than, equal to, or less than 1: that is, as $h$, the area in time 1 in the curve, is less than, equal to, or greater than $\sqrt{\mu}$, its value in the circle. So that if the area in a given time be *less* than that in a circle with the same force, we shall have Species I, II, or III; if the areas be *equal*, we have Species IV; if the area in the curve be *greater*, we have Species V.

In these two latter cases it is clear, that since the area is not less than it is in the circle when the radii vectors are the same, the velocity will be greater than it is in a circle. In the three first cases we may thus compare those velocities.

In the circle whose radius is $r$, since $r^2 . \text{velocity}^2=h^2=\mu$, we have $\text{velocity}^2=\frac{\mu}{r^2}=\mu u^2$*.

In the curve, $(\text{velocity})^2=h^2\left(u^2+\frac{du^2}{dv^2}\right)$.

$$\text{But } u=C\epsilon^{\gamma v}+C'\epsilon^{-\gamma v},$$

$$\left(\frac{du}{dv}\right)^2=\{\gamma C\epsilon^{\gamma v}-\gamma C'\epsilon^{-\gamma v}\}^2$$

$$=\gamma^2u^2-4\gamma^2CC'.$$

* This is also the velocity from an infinite distance.

Hence, (velocity$^2$) $= h^2(1+\gamma^2)u^2 - 4\gamma^2 CC'$

$= \mu u^2 - 4\gamma^2 CC'$; since $h^2(1+\gamma^2) = \mu$.

If the velocity be *less* than that in a circle, we have $CC'$ negative, and therefore the curve is Species I. If the velocity be *equal* to that in a circle, we have Species II. If the velocity in the curve be *greater*, we have Species III.

If the force be repulsive, the equation will resemble the one for Species V, and the curve, which we may call Species VI, will be as in fig. 19.

44. Prop. *To find the time of describing any portion of the curves described in Art. 42.*

In Species I, if we suppose the angle $v$ to be measured from the apse, and consequently $\alpha = 0$, we shall have

$$h = r^2 \frac{dv}{dt} = \frac{1}{u^2}\frac{dv}{dt} = \frac{1}{c^2}\frac{1}{\{\epsilon^{\gamma v} + \epsilon^{-\gamma v}\}^2}\frac{dv}{dt}$$

$$= \frac{1}{c^2}\cdot\frac{\epsilon^{2\gamma v}}{\{\epsilon^{2\gamma v}+1\}^2}\frac{dv}{dt};$$

$$\therefore\ ht = C - \frac{1}{2c^2\gamma}\cdot\frac{1}{\epsilon^{2\gamma v}+1}.$$

We may suppose the time to begin when $v = 0$: on this supposition we have

$$t = \frac{1}{2c^2\gamma h}\left\{\frac{1}{2} - \frac{1}{\epsilon^{2\gamma v}+1}\right\} = \frac{a^2}{\gamma h}\cdot\frac{\epsilon^{2\gamma v}-1}{\epsilon^{2\gamma v}+1};\ \text{if } a = SA = \frac{1}{2c}.$$

Similarly, we should find

in Species II, $t = \dfrac{a^2}{2\gamma h}\{1 - \epsilon^{-2\gamma v}\} = \dfrac{a^2}{\cot.\beta\sqrt{\mu}}\{1-\epsilon^{-2\gamma v}\}$,

$v$ and $t$ being measured from the point where the radius vector $= a$.

In Species III, $t = \dfrac{2a^2}{\gamma h}\dfrac{1}{\epsilon^{2\gamma v}-1}$, if $a = \dfrac{1}{2c}$; $v(= ASP)$, being measured from $SA$, and $t$ being the time from $P$ to the centre.

In Species IV, $t = \frac{a^2}{hv} = \frac{a^2}{\sqrt{\mu} . v}$; $v$ being measured from $SA$, and $t$ being the time to the centre.

In Species V, $t = \frac{a^2}{\gamma h} \tan. \gamma v$; $v$ and $t$ being measured from the apse.

COR. 1. In order to find the time $\tau$ of describing a given angle $\delta$, we must take the value of $t$ between the values $v$ and $v + \delta$; we shall thus have in Species I,

$$h\tau = \frac{1}{2c^2\gamma}\left\{\frac{1}{\epsilon^{2\gamma v} + 1} - \frac{1}{\epsilon^{2\gamma(v+\delta)} + 1}\right\}$$

$$= \frac{1}{2c^2\gamma} . \frac{\epsilon^{2\gamma(v+\delta)} - \epsilon^{2\gamma v}}{(\epsilon^{2\gamma v} + 1)(\epsilon^{2\gamma(v+\delta)} + 1)}$$

$$= \frac{1}{2c^2\gamma} . \frac{\epsilon^{\gamma\delta} - \epsilon^{-\gamma\delta}}{\{\epsilon^{\gamma v} + \epsilon^{-\gamma v}\}\{\epsilon^{\gamma(v+\delta)} + \epsilon^{-\gamma(v+\delta)}\}}$$

$$= \frac{\epsilon^{\gamma\delta} - \epsilon^{-\gamma\delta}}{2\gamma} . r_1 r_2;$$

$r_1$ and $r_2$ being the radii at the beginning and end of the given angle.

Similarly,

in Species II, $h\tau = \frac{(\epsilon^{\gamma\delta} - \epsilon^{-\gamma\delta})}{2\gamma} r_1 r_2$,

in Species III, $h\tau = \frac{(\epsilon^{\gamma\delta} - \epsilon^{-\gamma\delta})}{2\gamma} r_1 r_2$,

in Species IV, $h\tau = \delta . r_1 r_2$,

in Species V, $h\tau = \frac{\sin. \gamma\delta}{\gamma} r_1 r_2$,

COR. 2. In all the cases, the times of successive revolutions in the same spiral are as the extreme radii.

Let a straight line *SRQP*, fig. 14, drawn from *S*, cut the spiral successively in *R*, *Q*, *P*: then, since in this case $\delta = 2\pi$ is constant, we have

time from $P$ to $Q$ : time from $Q$ to $R$ :: $SP.SQ : SQ.SR$
:: $SP : SR$.

## *Sect. VI.* The Inverse Fifth Power. Asymptotic Circles.

45. Prop. *Let the force be inversely as the 5th power of the distance, or* $\mathrm{P} = \frac{\mu}{r^5} = u^5$.

Therefore, by (*d*), $\frac{d^2u}{dv^2} + u - \frac{\mu u^3}{h^2} = 0$;

and multiplying by $2\frac{du}{dv}$, and integrating with regard to $v$,

$$\frac{du^2}{dv^2} + u^2 - \frac{\mu u^4}{2h^2} = C;$$

$C$ being an arbitrary constant; hence,

$$\frac{du^2}{dv^2} = C - u^2 + \frac{\mu u^4}{2h^2}.$$

This equation cannot be integrated generally by the common methods.

When the right-hand member is a square, it becomes simple; that is, if 4 times the product of the extreme terms be equal to the square of the middle term;

$$\text{if } \frac{2C\mu}{h^2} = 1; \quad \text{if } C = \frac{h^2}{2\mu}.$$

On this supposition,

$$\frac{du}{dv} = \pm\frac{1}{\sqrt{2}}.\left\{\frac{h}{\sqrt{\mu}} - \frac{u^2\sqrt{\mu}}{h}\right\} = \pm\frac{\sqrt{\mu}}{h\sqrt{2}}\left\{\frac{h^2}{\mu} - u^2\right\};$$

and we shall have two different Species as we take the + or the − sign.

In the first case, $\dfrac{\dfrac{2h}{\sqrt{\mu}} \cdot \dfrac{du}{dv}}{\dfrac{h^2}{\mu} - u^2} = \sqrt{2}$,

$$\therefore \; \mathrm{l} \, \frac{\dfrac{h}{\sqrt{\mu}} + u}{\dfrac{h}{\sqrt{\mu}} - u} = \sqrt{2} \, . \, (v - a) \, ;$$

$a$ being the value of $v$, when $u = 0$.

When $v = a$, $r$ is infinite; as $u$ increases and $r$ decreases, $v$ increases; and when $u = \dfrac{h}{\sqrt{\mu}}$, or $r = \dfrac{\sqrt{\mu}}{h}$, $v$ is infinite. Hence, the curve, fig. 20, has what may be called an *asymptotic circle* with radius $SA = \dfrac{\sqrt{\mu}}{h}$, to which circle it perpetually approximates, but which it never actually reaches.

We have $\dfrac{1}{p^2} = u^2 + \dfrac{du^2}{dv^2} = u^2 + \dfrac{h^2}{2\mu} - u^2 + \dfrac{\mu u^4}{2h^2} = \dfrac{h^2}{2\mu} + \dfrac{\mu u^4}{2h^2}$;

and when $r$ is infinite, or $u = 0$, $p = \dfrac{\sqrt{(2\mu)}}{h}$; which is $SB$, the distance of the asymptote $BZ$ from $SA$.

In the second case, we have

$$\frac{\dfrac{2h}{\sqrt{\mu}} \, \dfrac{du}{dv}}{u^2 - \dfrac{h^2}{\mu}} = \sqrt{2},$$

$$\therefore \; \mathrm{l} \, \frac{u - \dfrac{h}{\sqrt{\mu}}}{u + \dfrac{h}{\sqrt{\mu}}} = \sqrt{2} \, . \, (v - a),$$

$a$ being the value of $v$ when $u$ is infinite.

Hence, $$\text{l}\,\frac{u+\dfrac{h}{\sqrt{\mu}}}{u-\dfrac{h}{\sqrt{\mu}}}=\sqrt{2}\,.\,(a-v).$$

When $v=a$, $u$ is infinite, and $r=0$; as $u$ decreases, or $r$ increases, $a-v$ also increases; and when

$$u=\frac{h}{\sqrt{\mu}},\ \text{or}\ r=\frac{\sqrt{\mu}}{h},$$

$a-v$ is infinite. Hence, the curve has, in this case also, an asymptotic circle, and is situated within it, as it was before without it. See fig. 21 $SQ$ comes to $SA$ when $v=0$; and

$$ASP=a-v.$$

Cor. 1. We shall now compare the velocity with that in a circle.

In a circle with radius $=r$, velocity$^2=Pr$, (see Art. 20.) $=\frac{\mu}{r^4}=\mu u^4$.

In the curve, velocity$^2=h^2\left(u^2+\frac{du^2}{dv^2}\right)$

$$=\frac{h^4}{2\mu}+\frac{\mu u^4}{2}.$$

Now when $r=SA$, or $u^2=\frac{h^2}{\mu}$,

$$\text{velocity}^2\text{ in curve}=\frac{\mu u^4}{2}+\frac{\mu u^4}{2}=\mu u^4=\text{velocity}^2\text{ in circle};$$

which it manifestly should be, because as the radius approximates to $SA$, the motion approximates to circular motion.

In the first case $u$ is always less than $\frac{h}{\sqrt{\mu}}$, and hence the velocity is always greater than that in a circle.

In the second case $u$ is always greater than this value, and the velocity is less than that in a circle at the same distance.

COR. 2. To find the velocity, so that one of these curves may be described.

Let, at any point $P$, the angle $SPY = \beta$, $SY$ being a perpendicular on the tangent. Therefore $h^2 = \text{velocity}^2 . SY^2 = \text{velocity}^2 . r^2 \sin.^2 \beta$.

Now let the velocity be $\epsilon$ times that in a circle at the distance $SP$: that is, $\text{velocity}^2 = \epsilon^2 \mu u^4$: hence,

$$\epsilon^2 \mu u^4 = \frac{h^4}{2\mu} + \frac{\mu u^4}{2}; \;\therefore (2\epsilon^2 - 1)\mu^2 u^4 = h^4;$$

$$\text{but } h^2 = \epsilon^2 \mu u^4 . \frac{\sin.^2\beta}{u^2}; \;\therefore \sin.^2\beta = \frac{h^2}{\epsilon^2 \mu u^2} = \frac{\sqrt{(2\epsilon^2 - 1)}}{\epsilon^2}.$$

Hence, if $\epsilon$ be given, we can find $\sin.^2\beta$; and hence, the direction in which the body must be projected to describe the curve. It will belong to the first or second Species, as $\epsilon$ is greater or less than 1.

$$\text{Also } \epsilon^2 \sin.^2\beta = \sqrt{(2\epsilon^2 - 1)}; \; \epsilon^4 \sin.^4\beta - 2\epsilon^2 + \frac{1}{\sin.^4\beta} = \frac{1}{\sin.^4\beta} - 1;$$

$$\therefore \epsilon^2 \sin.^2\beta - \frac{1}{\sin.^2\beta} = \pm \frac{\sqrt{(1 - \sin.^4\beta)}}{\sin.^2\beta}; \; \epsilon^2 = \frac{1 \pm \sqrt{(1 - \sin.^4\beta)}}{\sin^4\beta}:$$

and the first or second curve will be described, as we take the upper or the lower sign.

COR. 3. By equating the values of $ASP$ in the two species, we should find for the same angle $ASP$, fig. 20, $SP^1 . SP = SA^2$.

## *Sect VII.* INVERSE $n^{\text{th}}$ POWER. TRAJECTORY WITH VELOCITY ACQUIRED FROM AN INFINITE DISTANCE.

46. PROP. *Let the force vary inversely as any power of the distance, or* $\mathrm{P = mu^n}$.

$$\text{Therefore } \frac{d^2u}{dv^2} + u - \frac{\mu u^{n-2}}{h^2} = 0;$$

multiplying by $2\dfrac{du}{dv}$, and integrating,

$$\frac{du^2}{dv^2} + u^2 - \frac{2\mu u^{n-1}}{(n-1)h^2} = C;$$

$$\text{whence, } \frac{1}{\sqrt{\left\{C - u^2 + \dfrac{2\mu u^{n-1}}{(n-1)h^2}\right\}}} = \frac{dv}{du};$$

and if the expression on the first side be integrable, we can find the relation between $u$ and $v$.

To find the time, we have

$$\frac{dt}{du} = \frac{dt}{dv} \cdot \frac{dv}{du} = \frac{1}{hu^2} \cdot \frac{dv}{du} = \frac{1}{hu^2\sqrt{\left\{C - u^2 + \dfrac{2\mu u^{n-1}}{(n-1)h^2}\right\}}}.$$

The quantity $C$ will depend upon the velocity, and will be known, if we know the velocity for a given point; which may be called *the velocity of projection*, if we consider this point as the beginning of the motion. For we have

$$\text{velocity}^2 = h^2\left(u^2 + \frac{du^2}{dv^2}\right) = \frac{2\mu u^{n-1}}{n-1} + h^2C,$$

and if, when $u = a$, we have velocity $= V$,

$$V^2 = \frac{2\mu a^{n-1}}{n-1} + h^2C; \text{ whence } C \text{ is known.}$$

It will be convenient to compare the velocity with that acquired by falling from an infinite distance. Let $q$ be the velocity acquired by falling through any space towards the centre. Therefore

$$q\frac{dq}{dr} = -\frac{\mu}{r^n},$$

$$q^2 = \frac{2\mu}{(n-1)r^{n-1}} + \text{const.};$$

and if $q$ be the velocity acquired from infinity,

$$\text{const.} = 0,\ q^2 = \frac{2\mu}{(n-1)r^{n-1}} = \frac{2\mu u^{n-1}}{n-1}.$$

Hence, if at the point of projection, where $u = a$, the velocity be $\epsilon$ times that from infinity, we have

$$\epsilon^2 . \frac{2\mu a^{n-1}}{n-1} = \frac{2\mu a^{n-1}}{n-1} + h^2 C;$$

$$\therefore\ h^2 C = (\epsilon^2 - 1) . \frac{2\mu a^{n-1}}{n-1}.$$

Cor. At the apsides we have $\frac{du}{dv} = 0$;

$$\therefore\ C - u^2 + \frac{2\mu u^{n-1}}{(n-1)h^2} = 0;$$

or, putting for $C$ its value, and dividing,

$$(\epsilon^2 - 1)\, a^{n-1} - \frac{(n-1)h^2}{2\mu} . u^2 + u^{n-1} = 0.$$

This may have four roots possible, $\{$for instance, if $n = 5$, and $\frac{(n-1)^2 h^4}{4\mu^2} > 4(\epsilon^2 - 1)a^4$,$\}$ but only two of these give apsidal distances; in fact the other two are always negative.

47. Prop. *In the particular case where the velocity is equal to that from infinity, to find the curves described.*

If the velocity be at one point that from infinity, it will be so at all points. For, by Art. 17, Cor. 2, if the velocity be acquired in falling from a distance $a$,

$$v^2 = \int_r \frac{\mu}{r^n} = \frac{\mu}{n-1}\left\{\frac{1}{r^{n-1}} - \frac{1}{a^{n-1}}\right\},$$

and if $a$ be infinite, $v^2 = \frac{\mu}{n-1}\frac{1}{r^{n-1}}$ at all points.

Here, in the expression of last Article, $C = 0$, and we can integrate. For we have

$$1 = \frac{1}{\sqrt{\left\{\frac{2\mu u^{n-1}}{(n-1)h^2} - u^2\right\}}}\frac{du}{dv} = \frac{1}{u\sqrt{\left\{\frac{\mu u^{n-3}}{(n-1)h^2} - 1\right\}}}\frac{du}{dv}.$$

Let $\frac{2\mu u^{n-3}}{(n-1)h^2} = y^2$; $\therefore \frac{(n-3)}{u}\frac{du}{dv} = \frac{2}{y}\frac{dy}{dv}$; $\frac{du}{dv} = \frac{2}{n-3}\cdot\frac{u}{y}\cdot\frac{dy}{dv}$;

$$\text{therefore, } 1 = \frac{2}{n-3}\cdot\frac{1}{y\sqrt{(y^2-1)}}\frac{dy}{dv}:$$

and, integrating with respect to $v$,

$$v - a = \frac{2}{n-3}\sec.^{-1} y; \quad \frac{n-3}{2}(v-a) = \sec.^{-1} y = \cos.^{-1}\frac{1}{y};$$

$$\therefore \cos.\frac{n-3}{2}(v-a) = \frac{1}{y} = \frac{h\sqrt{(n-1)}}{\sqrt{(2\mu)}\cdot u^{\frac{n-3}{2}}}.$$

$$\text{Hence, if } n > 3; \quad \frac{h\sqrt{(n-1)}}{\sqrt{(2\mu)}}\cdot r^{\frac{n-3}{2}} = \cos.\frac{n-3}{2}(v-a).$$

$$\text{Similarly, if } n < 3; \quad \frac{h\sqrt{(n-1)}}{\sqrt{(2\mu)}}\,\frac{1}{r^{\frac{3-n}{2}}} = \cos.\frac{3-n}{2}(v-a).$$

In the first case, it is manifest that when the first side is $= 1$, or $r^{\frac{n-3}{2}} = \frac{\sqrt{(2\mu)}}{h\sqrt{(n-1)}}$, the figure has an apse. The curve is symmetrical on the two sides of this apse, and $r$ diminishes as $v - a$ increases. When $\frac{n-3}{2}(v-a) = \frac{\pi}{2}$, or $v - a = \frac{\pi}{n-3}$, we have $r = 0$, and the curve passes through the centre.

In the second case, $r$ increases as $v - a$ increases. When $\frac{3-n}{2}(v-a) = \frac{\pi}{2}$, or $v - a = \frac{\pi}{3-n}$, $r$ is infinite, and the curve is parallel to it. To find the nature of the

infinite branch, we have, (since in Art. 46, $C = 0$,)

$$\frac{1}{p^2} = u^2 + \frac{dv^2}{du^2} = \frac{2\mu u^{n-1}}{(n-1)h^2};$$

and when $r$ is infinite, $u = 0$, and $p$ is infinite; hence, the branch has no asymptote.

## *Sect. VIII.* INVERSE $n^{\text{th}}$ POWER.

### ASYMPTOTIC CIRCLES.

48. PROP. *Let the force vary inversely as any power of the distance; it is required to find the conditions requisite that the orbit may have an asymptotic circle.* See Art. 45.

$P = \mu u^n$, and as before in Art. 46.

$$\frac{dv}{du} = \frac{1}{\sqrt{\left\{C - u^2 + \dfrac{2\mu u^{n-1}}{(n-1)h^2}\right\}}},$$

$$v = \int_u \frac{1}{\sqrt{\left\{C - u^2 + \dfrac{2\mu u^{n-1}}{(n-1)h^2}\right\}}}.$$

Now, if the orbit have an asymptotic circle, of which the radius is $\dfrac{1}{c}$, it is manifest that the value of $v$, taken from any value of $u$ up to $u = c$, will be infinite: that is, the integral on the right-hand side must be infinite when $u = c$. Also, $u = c$ is necessarily a factor of the denominator, because when $u = c$, $\dfrac{du}{dv} = 0$, and therefore $C - u^2 + \dfrac{2\mu u^{n-1}}{(n-1)h^2} = 0$. But if the denominator has *two* factors $u - c$, the integral will be infinite for $u = c$. For in that case

$$v = \int_u \frac{1}{\sqrt{\{(u-c)^2 . Q\}}}.$$

$Q$ involving $u^{n-3}$, and inferior powers of $u$. And if we put $u = c + x$, it is manifest that $Q$ will become $A + Bx +$ &c. and

$$v = \int_z \frac{1}{\sqrt{\{z^2 . (A + Bz + \&c.)\}}}$$

$$= \int_z \frac{1}{z\sqrt{A}} \left\{1 - \frac{Bz}{2A} + \&c.\right\}$$

$$= \frac{1z}{\sqrt{A}} - \frac{Bz}{2A\sqrt{A}} + \&c.;$$

the other terms involving direct powers of $z$. Hence, when $u = c$, and $z = 0$, $v$ becomes infinite.

We shall therefore have an asymptotic circle if there be the factor $u - c$ twice in the denominator of $\frac{du}{dv}$; that is, if the equation

$$\frac{2\mu u^{n-1}}{(n-1)h^2} - u^2 + C = 0$$

have two roots $c$, $c$.

But in this case the equation

$$\frac{2\mu u^{n-2}}{h^2} - 2u = 0, \text{ has one of these roots; therefore}$$

$$\frac{2\mu c^{n-1}}{(n-1)h^2} - c^2 + C = 0, \text{ or } C = c^2 - \frac{2\mu c^{n-1}}{(n-1)h^2},$$

and $\frac{2\mu c^{n-2}}{h^2} - 2c = 0$, or $h^2 = \mu c^{n-3}$.

Now, in the curve, velocity$^2 = h^2 \left(\frac{du^2}{dv^2} + u^2\right)$

$$\text{(by Art. 46,)} \quad = h^2 \left(\frac{2\mu u^{n-1}}{(n-1)h^2} + C\right);$$

$$\text{(putting for } C \text{ its value,)} = \frac{2\mu u^{n-1}}{n-1} + c^2h^2 - \frac{2\mu c^{n-1}}{(n-1)};$$

(putting for $h$ its value,) $= \dfrac{2\mu u^{n-1}}{n-1} + \mu c^{n-1} - \dfrac{2\mu c^{n-1}}{n-1}$*;

$$= \frac{\mu}{n-1}\left\{2u^{n-1} + (n-3)\,c^{n-1}\right\}.$$

Let, at any point, the value of $u$ be $b$, and the angle $SPY$, fig. 21, which the tangent makes with the radius vector, $\beta$; and suppose that at this point the velocity is $\epsilon$ times that in a circle. Now, in a circle

$$\text{velocity}^2 = \text{force} \times \text{radius (Art. 27.)} = \mu u^n \cdot \frac{1}{u} = \mu u^{n-1}.$$

And when $u = b$, velocity$^2$ in circle $= \mu b^{n-1}$, and velocity$^2$ in curve $= \epsilon^2 \mu b^{n-1}$.

$$\text{Hence, } \epsilon^2 \mu b^{n-1} = \frac{\mu}{n-1}\left\{2b^{n-1} + (n-3)c^{n-1}\right\};$$

$$\text{and } \left\{(n-1)\epsilon^2 - 2\right\} b^{n-1} = (n-3)c^{n-1};$$

$$\therefore\; c = b\left\{\frac{(n-1)\epsilon^2 - 2}{n-3}\right\}^{\frac{1}{n-1}}.$$

$$\text{Also, } h^2 = \frac{1}{b^2}\cdot \text{velocity}^2 . \sin.^2\beta = \frac{1}{b^2}\cdot \epsilon^2 \mu b^{n-1} . \sin.^2\beta$$

$$= \epsilon^2 . \mu b^{n-3} \sin.^2\beta.$$

But $h^2 = \mu c^{n-3}$;

$$\therefore\; c^{n-3} = \epsilon^2 b^{n-3} . \sin.^2\beta; \text{ and } c = b\,.\,(\epsilon\,.\sin.\beta)^{\frac{2}{n-1}}.$$

$$\text{Therefore } (\epsilon \sin.\beta)^{\frac{2}{n-3}} = \left\{\frac{(n-1)\epsilon^2 - 2}{n-3}\right\}^{\frac{1}{n-1}}.$$

$$\text{And } \sin.\beta = \frac{1}{\epsilon}\cdot\left\{\frac{(n-1)\epsilon^2 - 2}{n-3}\right\}^{\frac{n-3}{2n-2}},$$

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* Hence, when $u = c$, velocity$^2 = \mu c^{n-1} =$ velocity$^2$ in a circle; as it manifestly should be.

which gives the relation between the velocity and the direction of projection, in order that the curve may have an asymptotic circle.

The radius $\left(=\frac{1}{c}\right)$, of the circle, is easily found by the preceding formulæ. If $\frac{1}{b}$ is greater than $\frac{1}{c}$, the circle is an interior one, as in fig. 20: that is,

$$\text{if } \frac{(n-1)\epsilon^2-2}{n-3} > 1,$$

$$\text{if } (n-1)\epsilon^2 - 2 > n-3,$$

$$\text{if } \epsilon^2 > 1, \text{ or if } \epsilon > 1.$$

If on the contrary $\epsilon$ be less than 1, the circle is exterior to the curve, as in fig. 21.

It is clear that we must have $n > 3$.

In nearly the same way we may find the conditions requisite for the description of an orbit, with an asymptotic circle, when the force is represented by any function whatever of $u$.

### *Sect. IX.* Revolving Orbits.

49. Prop. *Let the force consist of two parts, one of which varies inversely as the square, and the other inversely as the cube, of the distance.*

$$P = \mu u^2 + \mu' u^3;$$

$$\therefore \frac{d^2u}{dv^2} + u - \frac{\mu}{h^2} - \frac{\mu' u}{h^2} = 0;$$

$$\text{or } \frac{d^2u}{dv^2} + \left(1 - \frac{\mu'}{h^2}\right)u - \frac{\mu}{h^2} = 0.$$

To integrate, let $\left(1 - \frac{\mu'}{h^2}\right)u - \frac{\mu}{h^2} = \left(1 - \frac{\mu'}{h^2}\right)w,$

$$\text{or } u = w + \frac{\mu}{h^2 - \mu'};$$

$$\therefore \frac{d^2w}{dv^2} + \left(1 - \frac{\mu}{h^2}\right)w = 0; \text{ or if } 1 - \frac{\mu'}{h^2} = \gamma^2,$$

$$\frac{d^2w}{dv^2} + \gamma^2 w = 0:$$

of which, by nearly the same process as in Art. 30, we shall find the integral to be

$$w = C_1 \cos.\gamma v + C_2 \sin.\gamma v;$$

$$\therefore u = C_1 \cos.\gamma v + C_2 \sin.\gamma v + \frac{\mu}{h^2 - \mu'}.$$

This may be transformed in exactly the same manner as in Art. 29; that is, let $a$ be the value of $v$, which makes $\frac{du}{dv} = 0$, then the value which gives $\gamma v = \pi + \gamma a$ will also make $\frac{du}{dv} = 0$; and if $\frac{1}{r}$, $\frac{1}{r''}$, be the values of $u$, corresponding to these values of $v$, we shall have

$$u = \frac{1}{r} = \tfrac{1}{2}\left\{\frac{1}{r'} - \frac{1}{r''}\right\}(\cos.\gamma v - a) + \tfrac{1}{2}\left\{\frac{1}{r'} + \frac{1}{r''}\right\};$$

which is the equation to the curve described, if $r'$ and $r''$ be positive.

This manifestly agrees with the equation to an ellipse, of which the focus is in the centre of forces, except in having $\gamma(v - a)$ instead of $v - a$. Hence, the curve may be thus described: if, in fig. 148, and 149, $Vp$ be an ellipse of which the focus is $S$; and, $Sp$ being any radius, if we take $VSP = \frac{VSp}{\gamma}$; so that, $VSP$ being $v - a$, we may have $VSp = \gamma(v - a)$; then $SP = Sp$ may be $r$, and the equation just found for $r$ will be satisfied; therefore the curve $VPB$ thus described will be the path of the body.

*VPB* will be without the ellipse *Vp*, fig. 148, if $\gamma$ be less than unity; that is, if $1 - \frac{\mu'}{h^2} < 1$, or if $\mu'$ be positive. If $\mu'$ be negative, or the force be $P = \mu u^2 - \mu' u^3$, the path described will be within the ellipse, as in fig. 149.

In both cases we shall have an apse *B* in the curve, corresponding to an apse *b* in the ellipse; at which point $VSb = \gamma \,.\, VSB$, and $SB = Sb$. Hence, since $VSB = \pi$, we have

$$VSB = \frac{VSb}{\gamma} = \frac{\pi}{\sqrt{\left(1 \pm \frac{\mu'}{h^2}\right)}}.$$

*VSB* is *the angle between the apsides.*

After describing an angle $BSV' = VSB$, the body will come again to an apse at $V'$, and so go on perpetually revolving about *S*; approaching to it, and receding from it alternately.

The *line of apsides SV* retains always the same position, when a body describes an ellipse as in Art. 30. In the case of the present problem, this line, which is at first in the position *SV*, fig. 148, 149, would, after one revolution, come into the position $SV'$; after a second, into the position $SV''$; and so on; the angles $VSV'$, $V'SV''$, &c. being equal. Hence, this line is said to *revolve* round *S*. If it revolve in the direction of the body's motion, as in fig. 148, it is said to move *in consequentia*, or to *progress;* if it move in the opposite direction, as in fig. 149, it is said to move in *antecedentia*, or to *regress.* It appears by what has preceded, that the first or the second of these cases will occur, as the part of the force, $\mu' u^3$, which varies inversely as the cube of the distance, is additive, or subtractive.

If $P = \mu u^2 + \mu' u^3$ so that we have $VSB = \dfrac{\pi}{\sqrt{\left(1 - \frac{\mu'}{h^2}\right)}}$

it is manifest that we must have $\frac{\mu'}{h^2} < 1$; and therefore $h^2 > \mu'$.

When $h^2 < \mu'$, the body will fall into the centre without coming to a second apse, as might be shewn by integrating the equation

$$\frac{d^2u}{dv^2} - \left(\frac{\mu'}{h^2} - 1\right) u - \frac{\mu}{h^2} = 0.$$

When $h^2 = \mu'$, $\dfrac{d^2u}{dv^2} - \dfrac{\mu}{h^2} = 0$; $\dfrac{du}{dv} = \dfrac{\mu}{h^2}(v - a)$;

$u + \dfrac{\mu}{h^2} \cdot \dfrac{(v-a)^2}{2} = c$; supposing that $u = c$ when $v = a$.

In this case the body approaches the centre by an indefinite number of revolutions.

50. Prop. *Let the force be represented by any function of the distance; it is required to find what value the angle between the apsides approximates to, when the orbit becomes very nearly a circle.*

It is manifest, that if we project a body perpendicularly to the radius vector, with a velocity a very little greater or less than the velocity in a circle for the same distance and force, the path of the body will not differ much from a circle. With many laws of force, the body will revolve perpetually between its greatest and least apsidal distances, as in last Prop. fig. 148, 149; and the angle between the apsides will depend both upon the velocity and the law of force. As, however, the velocity approaches more nearly to that in a circle, the angle between the apsides will tend nearer and nearer to a certain limit. This limit it can never reach, because when the velocity becomes accurately that in a circle, the two apsidal distances are equal, a circle is the curve described, and there is no longer, properly speaking, an angle between the apsides, as every point is an apse. But if we find this limiting angle, it may serve to indicate what is the angle between the apsides, when the difference of the higher and lower apsidal distances is small, but finite.

Let $P = u^2 \phi u$; where $\phi u$ is a function of $u$; so that $P$ may be any function whatever of $u$;

$$\therefore \text{ by } (d), \; \frac{d^2 u}{dv^2} + u - \frac{\phi u}{h^2} = 0.$$

Now at the point where the body is projected perpendicularly to the radius, let $u = c$; and for any other point let $u = c + x$, $x$ being small. Then

$$\phi u = \phi c + \phi' c \,.\, x + \phi'' c \,.\, \frac{x^2}{1\,.\,2} + \&c.$$

Also if $1 : 1 + \delta$ were the ratio of the velocity² to the velocity² in a circle at the point of projection, we should have in the circle, velocity² = force × radius (Art. 27) $= c^2 \phi c \,.\, \frac{1}{c} = c\phi c$;

$\therefore$ in the curve, velocity² $= \dfrac{c\phi c}{1 + \delta}$;

$$\text{hence in the curve, } h^2 = \frac{\text{velocity}^2}{c^2} = \frac{\phi c}{c(1 + \delta)};$$

therefore, substituting in the original equation, we have

$$\frac{d^2 x}{dv^2} + c + x - \frac{(1 + \delta) c}{\phi c} \left\{ \phi c + \phi' c \,.\, x + \phi'' c \,.\, \frac{x^2}{1\,.\,2} + \&c. \right\} = 0;$$

$$\text{or } \left.\begin{array}{l} \dfrac{d^2 x}{dv^2} + \left(1 - \dfrac{c\phi' c}{\phi c}\right) x - \dfrac{c\phi'' c}{\phi c} \,.\, \dfrac{x^2}{1\,.\,2} - \&c. \\ \qquad - c\delta - \dfrac{c\phi' c}{\phi c} x\delta - \&c. \end{array}\right\} = 0.$$

And when the orbit becomes indefinitely nearly a circle, $\delta$ becomes indefinitely small, as does $x$; and hence, $x^2$, $x\delta$, &c. may be omitted in comparison of $x$:

$$\text{hence, } \frac{d^2 x}{dv^2} + \left(1 - \frac{c\phi' c}{\phi c}\right) x - c\delta = 0.$$

If we make $1 - \dfrac{c\phi' c}{\phi c} = \gamma^2$, we shall have, as in Art. 29, for the integral of this equation,

$$x = C_1 \cos. \gamma v + C_2 \sin. \gamma v + \frac{c\delta}{\gamma^2};$$

and $u = c + x = C_1 \cos. \gamma v + C_2 \sin. \gamma v + c + \dfrac{c\delta}{\gamma^2}$:

which indicates the same kind of orbit as is described in the last problem. And here, as there, we shall have

$$A = \text{the angle between the apsides} = \frac{\pi}{\gamma} = \frac{\pi}{\sqrt{\left\{1 - \frac{c\phi' c}{\phi c}\right\}}}.$$

Ex. 1. Let the force vary inversely as any power of the distance,

$$P = \mu u^n = u^2 . \mu u^{n-2}; \quad \therefore \phi u = \mu u^{n-2}; \quad \phi' u = (n-2) \mu u^{n-3};$$

$$\therefore \gamma^2 = 1 - \frac{(n-2)\mu c^{n-2}}{\mu c^{n-2}} = 3 - n;$$

$$\therefore \text{the angle between the apsides} = \frac{\pi}{\sqrt{(3-n)}}.$$

When $n = -1$, $A = \dfrac{\pi}{2}$, which agrees with Art. 20;

when $n = 0$, $A = \dfrac{\pi}{\sqrt{3}}$;

when $n = 1$, $A = \dfrac{\pi}{\sqrt{2}}$;

when $n = 2$, $A = \pi$, which agrees with Art. 30;

when $n = 3$, $A$ is infinite;

and when $n > 3$, the expression is impossible. In fact, in this case, if the body leave one apse, it will never reach another; but will go off to infinity, if the velocity be greater than that in a circle, and fall to the centre if the velocity be less*.

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* This is also true if the velocity be not nearly equal to that in a circle, as might be shewn.

The Student will find an investigation of the angle between the apsides, in some cases when the orbit is not nearly circular, in the Transactions of the Cambridge Philosophical Society, Vol. I. Part I. p. 179.

If $n$ be a little greater than 2, the apsides progress slowly: thus the apse will advance about $3^0$ in one revolution, or $1\frac{1}{2}^0$ in a semi-revolution, if $n = 2\frac{4}{243}$.

Ex. 2. Let the force consist of two parts, each varying as any power of the distance;

$$P = \mu u^n + \mu' u^{n'} = u^2 (\mu u^{n-2} + \mu' u^{n'-2});$$

$$\therefore\ \phi u = \mu u^{n-2} + \mu' u^{n'-2},\ \phi' u = (n-2)\mu u^{n-3} + (n'-2)\mu' u^{n'-3};$$

$$\therefore\ \gamma^2 = 1 - \frac{(n-2)\mu c^{n-2} + (n'-2)\mu' c^{n'-2}}{\mu c^{n-2} + \mu' c^{n'-2}}$$

$$= \frac{(3-n)\mu c^{n-2} + (3-n')\mu' c^{n'-2}}{\mu c^{n-2} + \mu' c^{n'-2}};$$

whence $A = \frac{\pi}{\sqrt{\gamma}}$, is known.

Ex. 3. Let the force vary as the sine of the distance from the centre, the distance being considered as an arc.

Let $q$ be the distance, which, in this variation, is considered as a quadrant; and $\mu$ the force at that distance: then,

$$\sin.q : \sin.r :: \mu : \mu . \frac{\sin r}{\sin.q} = \text{force at distance } r: \text{ the}$$

sines being taken to such a radius that $q$ is a quadrant.

But, if the sines of the corresponding angles be taken to radius 1, they will be in the same ratio:

and $q : r :: \frac{\pi}{2} : \frac{\pi r}{2q}$, the angle corresponding to $r$;

$$\therefore \text{ force} = P = \mu . \sin. \frac{\pi r}{2q} = \mu . \sin. \frac{\pi}{2qu}$$

$$= u^2 . \frac{\mu}{u^2} . \sin. \frac{\pi}{2qu};$$

$$\therefore\ \phi u = \frac{\mu}{u^2} . \sin. \frac{\pi}{2qu},$$

$$\phi' u = -\frac{2\mu}{u^3} . \sin. \frac{\pi}{2qu} - \frac{\mu}{u^2} . \frac{\pi}{2qu^2} \cos. \frac{\pi}{2qu}.$$

Hence, $\gamma^2 = 1 - \frac{c\phi' c}{\phi c} = 3 + \frac{\pi}{2qc} \operatorname{cotan.} \frac{\pi}{2qc}$;

where $\frac{1}{c}$ is the radius of the circle to which the orbit approximates.

If we make $\frac{1}{c} = a$, we have

$$\gamma^2 = 3 + \frac{\pi a}{2q} \operatorname{cotan.} \frac{\pi a}{2q}.$$

If $a = 0$, $\gamma^2 = 4$, $\gamma = 2$.

If $a = \frac{1}{2}q$, $\gamma^2 = 3 + \frac{\pi}{4}$.

If $a = q$, $\gamma^2 = 3$.

The angle between the apsides varies from $\frac{\pi}{2}$ to $\frac{\pi}{\sqrt{3}}$ according to the different magnitudes of the circle described.

## Geometrical Investigation for Sect. IX.

(Newton, Book I. Sect. ix.)

51. (Newton, Prop. xliv.) Fig. 148. *If a body revolve in the orbit* VP *about the centre of force* C, *and if the angle* VCp *be taken always in a given ratio to* VCP, (G : F) *and* Cp *equal to* CP; *the orbit* Vp *may also be described by a force tending to* C, P *and* p *being always at corresponding points.*

For the area $VCP$ is proportional to the time (Introd. Prop. i.) But it is easily seen that the area $VCp$ is to the area $VCP$ in the given ratio ($G : F$). Hence the area $VCp$ is so described as to be proportional to the time. If therefore we find (by Prop. vi. Introd.) the force at $C$ by which the body may describe the orbit $Vp$, the body $p$ will move in the manner asserted in the Proposition.

52. (Newton, Prop. XLV.) *To find the difference of the forces by which the bodies* P *and* p *are retained in their orbits.*

Fig. 150. Let $P$, $p$ be corresponding points of the orbits, and also $K$, $n$. Take the angle $pCk = PCK$, and describe, with centre $C$, a circle $Kkn$. Draw $rkm$ perpendicular to $Cp$, and take $mr : kr :: G : F$.

The motions of the bodies at $P$ and $p$ may be resolved each into two parts, the first parts being in the directions $PC$, $pC$, respectively, the second parts perpendicular to these directions. The former parts are equal in the two cases, because $Cp$ is always equal to $CP$: the latter parts are always in the ratio $F : G$; for the velocities transverse to $CP$, $Cp$ will be in the *ultimate* ratio of the angles $VCP$, $VCp$. Hence, if $P$, $p$ were to proceed from $P$, $p$ with the velocities which they there have, and were both to be acted upon by the force which acts in $P$, they could still have equal velocities in the radii $PC$, $pC$, and velocities transverse to these radii which would be in the ratio $F : G$. Hence, while the former body come to $k$, the latter would come to $m$, because $pr = PR$, ($KR$ being perpendicular to $CP$) and $mr : kr :: G : F$.

But in order that the second body may describe the orbit $Vp$, it must be drawn to $n$, while the first body is brought to $k$. Hence, we must have, besides the force which acts on $P$, an additional force which shall draw the body through $mn$ in the time of describing $pn$.

If $f$ be this force, and $t$ the time of describing $PK$ or $pn$,

$$mn = \tfrac{1}{2}ft^2, \quad f = \frac{2mn}{t^2}.$$

Now if $mr$ produced meet the circle again in $s$, and if $mn$ meet it in $t$, we have $mn \,.\, mt = mk \,.\, ms$. But $mr = \dfrac{G}{F}kr$,

$$\text{whence } mk = \frac{G-F}{F}kr, \quad ms = \frac{G+F}{F}kr.$$

And $mt$ ultimately passes through the centre*, and therefore $=2Cp$. Hence, ultimately,

$$2Cp \, . \, mn = \frac{(G+F)(G-F)}{F^2}(kr)^2; \text{ whence } mn = \frac{G^2-F^2}{F^2} \cdot \frac{(kr)^2}{2Cp}.$$

Also if $A$ be the area in a unit of time, decribed by $P$, $At$ is equal to $PCK$ or $pCk$; that is, $At = \frac{Cp \, . \, kr}{2}$,

$$\text{whence } kr = \frac{2At}{Cp}; \quad \text{and hence}$$

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* If the sine of $nx$ be drawn, and if $mn$ meet $xC$ in $o$, it will easily be seen that we have

$$mr : \sin . nx :: Co + Cr : Co + \cos . nx,$$

$$\text{and } mr : mr - \sin . nx :: Co + Cr : Cr - \cos . nx;$$

$$\text{whence } Co + Cr = mr \frac{Cr - \cos . nx}{mr - \sin . nx}.$$

Let $xk = x$, and $\frac{G}{F} = \gamma$: also $Ck = 1$.

$$\text{Therefore } kr = \sin . x, \quad mr = \gamma \sin . x, \quad Cr = \cos . x,$$

$$\sin . nx = \sin . \gamma x, \quad \cos . nx = \cos . \gamma x.$$

$$\text{Hence, } Cr - \cos . nx = \cos . x - \cos . \gamma x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&\text{c.}$$

$$-1 + \frac{\gamma^2 x^2}{1.2} - \frac{\gamma^4 x^4}{1.2.3.4} + \&\text{c.} = (\gamma^2 - 1)\frac{x^2}{2} + \&\text{c.}$$

$$mr - \sin . nx = \gamma \sin . x - \sin . \gamma x = \gamma \left(x - \frac{x^3}{1.2.3} + \&\text{c.}\right) - \gamma x + \frac{\gamma^3 x^3}{1.2.3} - \&\text{c.}$$

$$= \gamma(\gamma^2 - 1)\frac{x^3}{6} - \&\text{c.}$$

$$mr = \gamma \sin . x = \gamma x - \frac{\gamma x^3}{1.2.3} + \&\text{c.}$$

Hence, by omitting the higher powers of $x$, because $x$ vanishes ultimately,

$$Co + Cr = \gamma x \, . \, \frac{\frac{x^2}{2}}{\frac{\gamma x^3}{6}} = 3.$$

And ultimately $Co + Cr = xo$, of which the value is rightly found by making $x$ to vanish. Therefore $xo = 3$, $Cx$ being $= 1$.

$$mn = \frac{G^2 - F^2}{F^2} \cdot \frac{2A^2 t^2}{Cp^3}; \quad \text{and } f = \frac{2mn}{t^2} = \frac{G^2 - F^2}{F^2} \cdot \frac{4A^2}{Cp^3},$$

hence the difference of forces is inversely as $Cp^3$.

COR. 1. If a body describe the circle $Kk$, describing the arc $\varkappa k$ in time $t$, the area in a unit of time is the same as in the orbit of $P$, for the areas $Cpk$, $C\varkappa k$, are ultimately equal. Hence, in this circle the force is (Introd. Prop. VI. Cor. 3.) $\frac{4A^2}{Cp^3}$. Hence, $f$ : force in circle with same angular velocity as $P$ :: $\frac{G^2 - F^2}{F^2}$ : 1.

COR. 2. If the orbit $VP$ be an ellipse of which $C$ is the focus, we have the force at $P = \frac{8A^2}{L \cdot CP^2}$ (Introd. Prop. XI.) $L$ being the latus rectum. Hence, the whole force at $p$

$$= \frac{8A^2}{L \cdot Cp^2} + \frac{G^2 - F^2}{F^2} \cdot \frac{4A^2}{Cp^2} = \frac{8A^2}{L \cdot F^2} \left\{ \frac{F^2}{Cp^2} + \frac{(G^2 - F^2) \cdot \frac{1}{2} L}{Cp^3} \right\};$$

and the force varies as $\frac{F^2 \cdot Cp + (G^2 - F^2) \cdot \frac{1}{2} L}{Cp^3}$.

COR. 3. If $VP$ be an ellipse of which $C$ is the centre, force on $P = \frac{4A^2 \cdot Cp}{a^2 b^2}$, $a$, $b$ being the semi-axis; hence force on $p$

$$= \frac{4A^2 \cdot Cp}{a^2 b^2} + \frac{G^2 - F^2}{F^2} \frac{4A^2}{Cp^3} = \frac{4A^2}{F^2} \cdot \left\{ \frac{F^2}{a^2 b^2} \cdot Cp + \frac{G^2 - F^2}{Cp^3} \right\}.$$

COR. 4. If $R$ = the radius of curvature at the apse $V$, the force in $P$'s orbit at $V$ is (Introd. Prop. VI. Cor. 3.) $\frac{4A^2}{CV^2 \cdot R}$; and therefore at $V$, $f = \frac{G^2 - F^2}{F^2} \cdot \frac{4A^2}{CV^2 \cdot R}$; but at other points

$$f = \text{force at } V \cdot \frac{CV^3}{Cp^3} = \frac{G^2 - F^2}{F^2} \frac{4A^2 \cdot CV}{R \cdot Cp^3};$$

and the whole force at $p$ = force at $P + \frac{G^2 - F^2}{F^2} \cdot \frac{4A^2 \cdot CV}{R \cdot Cp^3}$.

Cor. 5. If $G : F$ be a ratio of less inequality, so that $Vp$ falls within $VP$, the force on $p$ will be less than that on $P$. The same expressions as before will be true for the difference of these forces.

Cor. 6. Hence, knowing the force in $VP$, we can find the force in $Vp$; and hence from given orbits find others.

Cor. 7. If $VP$ be a straight line perpendicular to $CP$, the force at $P$ vanishes; and the force at $p = \frac{G^2 - F^2}{F^2} \cdot \frac{4A^2}{Cp^3}$, and varies inversely as $Cp^3$. Hence, if in Fig. 151, we take the angle $VCp$ in a given ratio to $VCP$, and $Cp = CP$, the locus of $p$ is an orbit which may be described by a force varying as the inverse cube. It may be shewn that this construction gives the orbits, Species 3 and 4 in Sect. v. of this Chapter.

53. (Newton, Prop. xlv.) *In orbits which are very nearly circular, it is required to find the angle between the apsides.*

In Cor. 2. of the last problem, the force may be made to approximate *ultimately* to any given law of force, by properly determining the ratio $G : F$. Hence, knowing the law of force, we may determine that ratio. And, when $P$ comes to an apse, $p$ also does; hence, knowing $G : F$, we determine the angle between the apsides of $p$'s orbit. The ultimate proportion of the terms expressing the law of force is found by putting $T - x$ for $Cp$, ($T$ being the apsidal distance $Cp$), and then making $x$ to vanish.

Ex. 1. Let the central force be uniform. It therefore is as $\frac{Cp^3}{Cp^3}$. But by Cor. 2, Art. 52, it is as

$$\frac{F^2 \cdot Cp + (G^2 - F^2) \cdot \frac{1}{2} L}{Cp^3}$$

Hence these two quantities must vary as each other; or, putting $T-x$ for $Cp$, and comparing the numerators,

$$F^2.(T-x)+(G^2-F^2).\tfrac{1}{2}L \text{ is as } (T-x)^3; \text{ that is,}$$

$$F^2.T+(G^2-F^2).\tfrac{1}{2}L-F^2x \text{ is as } T^3-3T^2x+\&c.,$$

$$\text{or as } \frac{T}{3}-x+\&c., \quad \text{or as } \frac{F^2.T}{3}-F^2x+\&c.$$

But, since the variable terms are ultimately equal, this cannot be the case, ultimately, when $x$ is very small, except the constant terms be equal. Therefore,

$$F^2.T+(G^2-F^2).\tfrac{1}{2}L=F^2\frac{T}{3}.$$

And ultimately, when the orbit becomes a circle, $\frac{1}{2}L$ is equal to $T$, and $G^2=\frac{F^2}{3}$ or $\frac{G}{F}=\frac{1}{\sqrt{3}}$.

Hence, the angle between the apsides of $p$'s orbit is

$$\frac{180^0}{\sqrt{3}}=103^0\ 55'\ 23''.$$

Ex. 2. Let the central force vary as $(Cp)^{n-3}$ or $\frac{Cp^n}{Cp^3}$. Hence, making $Cp=T-x$ as before, and comparing the numerators of the forces,

$$F^2(T-x)+(G^2-F^2)\tfrac{1}{2}L \text{ is as } (T-x)^n, \text{ or as } T^n-nT^{n-1}x+\&c.$$

$$\text{or } F^2T+(G^2-F^2)\tfrac{1}{2}L-F^2x \text{ as } \frac{F^2T}{n}-F^2x+\&c.$$

$$\text{whence } F^2T+(G^2-F^2)\tfrac{1}{2}L=\frac{F^2T}{n},$$

and ultimately, when $\frac{1}{2}L=T$,

$$G^2=\frac{F^2}{n},\ \frac{G}{F}=\frac{1}{\sqrt{n}}.$$

Thus, if the force vary inversely as the distance, or as $\frac{1}{Cp}$, this is as $\frac{Cp^2}{Cp^3}$, and $n = 2$. Hence, $\frac{G}{F} = \frac{1}{\sqrt{2}}$, and the angle between the apsides of $p$'s orbit is ultimately

$$\frac{180}{\sqrt{2}} = 127^0\ 18'\ 45'':$$

and a body acted on by such a force would ultimately, when the orbit was nearly circular, come alternately to a higher and lower apse, distant from each other by this angle.

If $n = \frac{1}{4}$, so that the force is as $\frac{1}{Cp^{3-\frac{1}{4}}}$ or $\frac{1}{Cp^{\frac{11}{4}}}$, $\frac{G}{F} = 2$, and the ultimate angle between the apsides of $Cp$'s orbit is $2 \times 180^0 = 360^0$; and the body would employ a complete revolution in descending from the higher to the lower apse; and again, a complete revolution in reascending to the higher, and so on.

Ex. 3. Let the force be as $b \,.\, Cp^{m-3} + c \,.\, Cp^{n-3}$. Hence the numerators are

$$F^2(T-x) + (G^2 - F^2)\tfrac{1}{2}L \text{ and } b(T-x)^m + c(T-x)^n,$$

$$\text{or } F^2T + (G^2 - F^2)\tfrac{1}{2}L - F^2x$$

$$\text{is as } b(T^m - mT^{m-1}x + \&c.) + c(T^n - nT^{n-1}x + \&c.),$$

$$\text{or as } bT^m + cT^n - (mbT^{m-1} + nT^{n-1})x + \&c.$$

Multiply the first side by $mbT^{m-1} + ncT^{n-1}$, the second by $F^2$, and the terms involving $x$ will be equal. Therefore the first terms must ultimately be so; and observing that ultimately $\frac{1}{2}L = T$, we have

$$(mbT^m + ncT^n)G^2 = (bT^m + cT^n)F^2,$$

$$\frac{G}{F} = \sqrt{\frac{bT^m + ncT^n}{mbT^m + ncT^n}}.$$

If we take $T$ for the unit of linear distance, $b$ and $c$ will be the values, at the apse $V$, of the two parts of the force which vary according to the indices $m - 3$ and $n - 3$. In this case

$$\frac{G}{F} = \sqrt{\frac{b + c}{mb + nc}}:$$

and the angle between the apsides of $p$'s orbit is

$$180^0 \sqrt{\frac{b + c}{mb + nc}}.$$

And in like manner, if $c$ were negative, it would be

$$180^0 \sqrt{\frac{b - c}{mb - nc}}.$$

Cor. 1. Hence, if the angle between the apsides be known, we can find the power of the distance according to which the force varies ultimately, in approaching the circular form of the orbit. If the angle from either apse to the same apse again be $m \cdot 360^0$, we shall have the force as

$$Cp^{\frac{1}{m^2}-3}, \text{ or as } \frac{1}{Cp^{3-\frac{1}{m^2}}}.$$

Hence, when the orbit has two apsides, the force cannot decrease in a higher inverse ratio than the cube. For in that case the angle between the apsides would be impossible. When the force varies in a higher inverse ratio than the cube, if the body proceed from an apse, and the distance increase at first, the body will go off to an infinite distance. If the distance diminish at first, the body will fall into the centre.

But if the force vary in a lower inverse ratio than the cube, or in a direct ratio, the body proceeding from an apse (as a higher apse), will come to another (as a lower,) and conversely.

And the smaller the angle described before meeting the second apse, the further does the law of force recede from the inverse cube, and *vice versâ*.

Thus, if the body move from the higher to the lower apse in revolutions 8, 4, 2, $1\frac{1}{2}$, the force will vary as

$$\frac{1}{Cp^{3-\frac{1}{64}}},\quad \frac{1}{Cp^{3-\frac{1}{16}}},\quad \frac{1}{Cp^{3-\frac{1}{4}}},\quad \frac{1}{Cp^{3-\frac{4}{9}}} \text{ respectively.}$$

If the body return to an apse at the end of every half revolution, $m = 1$, force $\propto \frac{1}{Cp^2}$.

If the body return to the same apse in $\frac{3}{4}$, $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{4}$ of a revolution, the force varies as

$$\frac{1}{Cp^{3-\frac{16}{9}}},\quad \frac{1}{Cp^{3-\frac{9}{4}}},\quad Cp^{9-3},\quad Cp^{16-3} \text{ respectively;}$$

or as $\frac{1}{Cp^{\frac{11}{9}}}$, $\frac{1}{Cp^{\frac{3}{4}}}$, $Cp^6$, $Cp^{13}$.

If the body return to the same apse in $363^0$, $m = \frac{363}{360} = \frac{121}{120}$. Hence, the force is as

$$\frac{1}{Cp^{3-(\frac{120}{121})^2}},\quad \text{or as } \frac{1}{Cp^{\frac{29523}{14641}}},\quad \text{or as } \frac{1}{Cp^{2\frac{4}{243}}}:$$

the force decreases very nearly as the square of the distance; the ratio being a little higher.

Cor. 2. Hence, if a body revolving in an ellipse by a force in the focus, varying inversely as the square of the distance, be affected also by an extraneous force, we can find the motion of the apsides by Ex. 3.

Thus, let the extraneous force be as the distance, and $= c \cdot Cp$, the principal force being $\frac{1}{Cp^2}$. Then, in Ex. 3,

$$b = 1, \quad m = 1, \quad n = 4.$$

Hence, the angle from higher to lower apse is

$$180\sqrt{\frac{1-c}{1-4c}}, \quad T \text{ being } 1.$$

Suppose that at the apse where $T = 1$, the extraneous force is $\frac{100}{35745}$ of the principal force. Then the angle

$$= 180\sqrt{\frac{35645}{35345}} = 180.7623 = 180^{\circ}\ 45'\ 44''.$$

Hence, a revolution from an apse to the same apse would occupy $361^{\circ}\ 31'\ 28''$; and in a revolution the apse itself advances $1^{\circ}\ 31'\ 28''$.

The above fraction $\frac{100}{35745}$ expresses nearly the proportion of the *radial* disturbing force which acts upon the Moon. But the motion of the Moon's apse is nearly $3^{\circ}$, or double that above found. The difference arises from the action of the transverse disturbing force, which also affects the motion of the apse, in the case of the Moon's orbit.

# CHAP. IV.

## ON THE MOTION OF SEVERAL POINTS.

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54. We have hitherto investigated the motions of bodies attracted towards fixed centres. Such however is not the kind of attraction which we generally find in nature. In the systems to which we have mainly to apply our dynamical reasonings, the attractions which operate upon some of the moving bodies are directed towards other bodies; and these are themselves in motion. The actions which two bodies exercise upon each other are mutual and equal, and the attracting body is thus acted upon by the same force (that is the same pressure) as the body attracted. Thus both move; and the effect of this mutual action is, as will be seen, to make them move about their common centre of gravity. If there be several bodies, which either all attract and are attracted by a single body, or all attract each other, these also will move in such a manner that the common centre of gravity will either remain at rest or move uniformly in a straight line. We shall now consider the cases of such motions.

We shall in the first place investigate some of the most important cases of the motion of several bodies, by methods independent of the general rectangular equations of motion. Such modes of investigation are in some respects simpler than those derived from the general equations; and they were originally employed by Newton to deduce the laws and values of some of the largest perturbations arising from the mutual attractions of the bodies of the solar system. The general analytical method is required for a more exact calculation of the perturbations; and we shall point out the preparatory steps of this application of the formulæ given in the preceding part of the work.

The *accelerating forces* on two bodies rising from their mutual action are inversely as the masses of the bodies.

For if $M$ be the mutual action or pressure which urges the two bodies towards each other, $T$, $P$ the bodies, by the third law of motion the accelerating forces are as $\frac{M}{T}$, $\frac{M}{P}$, that is, inversely as $T : P$.

The accelerating force of a body $T$ on another body $P$ is independent of the mass of $P$. For if a body $T$ exert a certain accelerating force upon a particle $p$, it will exert an equal accelerating force on another equal particle $p'$; and therefore if we suppose $p$ and $p'$ to be joined, the accelerating force on $p + p'$, or $2p$ will still be the same; and similarly for $3p$, $4p \ldots np$, whatever be $n$.

Hence the moving force of a given body $T$ on $P$ at a given distance is proportional to $P$: for the moving force is as $P$'s accelerating force.

## *Sect. I.* The Motion of two Bodies.

55. Prop. *If two bodies move, acted on by their mutual attraction, the centre of gravity will either remain at rest or move uniformly in a straight line.*

Case. 1. Fig. 152. Let the centre of gravity $C$ be at rest at any moment. Let $M$, $N$; $P$, $S$; be the positions of the bodies at proximate successive times; then

$$CS : CP :: P : S :: CN : CM,$$

therefore, $MCP$, $NCQ$ are similar figures, and $MP$ is ultimately parallel to $NQ$. Also the velocities at $M$, $N$ are as $MP$, $NS$, or as $CM$, $CN$. And in the next small interval of time the bodies would go on describing, with these velocities, $PR = PM$, and $SV = SN$, which also would be as $CM$ and $CN$. But the mutual attraction acts towards $C$, and draws the bodies through spaces $RQ$, $VT$, proportional to the accelerating forces; that is, to $\frac{M}{P}$ and $\frac{M}{S}$, or to $S$ and $P$, or to

*CM*, *CN*. Therefore, the figures *CPRQ*, *CSVT*, will still be similar, and

$$CQ : CT :: CM : CN :: S : P;$$

and *C* will be the centre of gravity, when the bodies come to *Q*, *T*; and therefore this centre is still at rest. And in like manner it might be shewn to be at rest after any number of such intervals. And when the number of such intervals is increased and their magnitude diminished indefinitely, the motion so represented approximates indefinitely to the real curvilinear motion of the bodies. Therefore in this motion the centre of gravity will be at rest.

Case 2. If the centre of gravity be in motion, we may suppose the whole space in which the bodies are, to have a motion given it in a direction opposite to that in which the centre of gravity moves, and with an equal velocity. This new motion will not affect the forces and relative motions of the bodies, as appears by the second law of motion. But this supposition will reduce the centre of gravity to absolute rest; and therefore the proof of the first law is here applicable.

Cor. The motion of each body about the centre of gravity is similar to the relative motion of each body about the other. (Newton, Book I. Prop. lvii.)

For the distance of each body from the other is always in a given ratio to its distance from the centre of gravity. And the direction of each body from the other is always in the same line as the direction of the first from the centre of gravity. Hence the curve which each describes about the other, is similar to the curve which it describes about the centre of gravity; and the motion in this curve also similar to the motion about the centre of gravity.

(Newton, Book I. Prop. lviii.)

56. Prop. *If two bodies attract each other, and revolve about their centre of gravity, so that each describes a relative orbit about the other; if one of them be supposed to become*

*fixed and exert the same force as before, and the other revolve about it; the last may be made to describe an* absolute *orbit similar and equal to the relative orbit of the former supposition.*

Fig. 152. Let the bodies $S$, $P$ revolve about the centre of gravity $C$, going from $S$ to $T$ and from $P$ to $Q$. Let $sp$, $sq$ be taken parallel and equal to $SP$, $TQ$; then the curve $pq$ will be similar to $PQ$, and equal to the corresponding part of the relative orbit of $P$ about $S$.

Let there be placed in $s$ and $p$, bodies similar and equal to $S$ and $P$ respectively, and let $s$ be fixed and $p$ revolve about it. Also let $PR$, $pr$ be tangents to the curves in $P$ and $p$, $QR$, $qr$ subtenses parallel to $PS$, $ps$. Since the two figures are similar, we shall have

$$QR : qr :: CP : sp :: CP : SP :: S : S+P.$$

Let $T$ be the time of describing $PQ$, and $t$ the time of describing $pq$. The forces which act on $P$, $p$ are equal, being in both cases the mutual attraction of $S$ and $P$. Hence the effects of the forces are as the squares of the times (Lemma x), and their effects are $QR$, $qr$, (by the second law of motion). Therefore,

$$T^2 : t^2 :: QR : qr :: S : S+P; \quad T : t :: \sqrt{S} : \sqrt{S+P}.$$

$$\text{Hence vel. of } P : \text{vel. of } p :: \frac{PQ}{T} : \frac{pq}{t} :: \frac{PQ}{pq} : \frac{T}{t}$$

$$:: \frac{CP}{sp} : \frac{T}{t} :: \frac{S}{S+P} : \frac{\sqrt{S}}{\sqrt{S+P}} :: \sqrt{S} : \sqrt{S+P}.$$

And if $P$ and $p$ move so that this proportion of the velocities obtains, they will arrive at $Q$, $q$ in the times $T$, $t$, having moved in a similar manner. They will therefore at $Q$, $q$ have velocities in the same proportion as before; and will go on again describing similar arcs, in times which are in the ratio $\sqrt{S} : \sqrt{S+P}$; and so on perpetually. Therefore the orbit of $p$ round $s$ will be described in a manner similar to that in which the relative orbit of $P$ round $S$ is described.

COR. 1. Hence if two bodies attract each other with forces varying directly as the distance, the bodies will describe about the centre of gravity and about each other, ellipses, of which those points are the centres: and conversely, if such figures are described, the forces vary directly as the distances.

COR. 2. And two bodies which attract each other with forces varying inversely as the square of the distance, describe about their centre of gravity, and about each other, conic sections, of which those points are the foci: and conversely if such orbits are described, the forces vary inversely as the square of the distance.

COR. 3. Any two bodies which revolve about their common centre of gravity by their mutual attraction, describe about that centre and about each other areas which are proportional to the times.

(NEWTON, Book I. Prop. LIX.)

57. PROP. *In the last Proposition, the periodic time in the absolute orbit of* P *round* S, *is to the periodic time of* P *or* S *round* C *in the ratio* $\sqrt{S+P} : \sqrt{S}$.

For by the last Proposition this is the ratio of the time of describing similar arcs; and therefore of describing the whole orbits.

COR. 1. Suppose two systems of two bodies, $S$, $P$ and $T$, $Q$, describing similar orbits; and let the attractive forces of $S$ on $P$ and $T$ on $Q$ be as the masses of $S$ and $T$, and inversely as the square of the distance.

Let $p$ and $q$ be the periodic times of $P$, $S$, and of $Q$, $T$, round their centres of gravity respectively. Then the periodic time of $P$ round $S$ at rest, at the same distance as before, is $p\dfrac{\sqrt{S+P}}{\sqrt{S}}$ by this Proposition; and similarly $q\dfrac{\sqrt{T+Q}}{\sqrt{Q}}$ is the period of $Q$ round $T$ at rest. But in the similar orbits the forces are as homologous lines directly and the square of the period inversely, (Introd. Prop. IV. Cor. 8.) Hence, we have

force of $S$ at distance $SP$ : force of $T$ at distance $TQ$ ::

$$\frac{SP}{p^2} \cdot \frac{S}{S+P} : \frac{TQ}{q^2} \cdot \frac{T}{T+Q}.$$

Or, since $S$ and $T$ are as the absolute attractive forces of $S$ and $T$;

$$\frac{S}{SP^2} : \frac{T}{TQ^2} :: \frac{SP}{p^2} \cdot \frac{S}{S+P} : \frac{TQ}{q^2} \cdot \frac{T}{T+Q},$$

$$\text{whence } p^2 : q^2 :: \frac{SP^3}{S+P} : \frac{TQ^3}{T+Q}.$$

COR. 2. If $T$ revolve round $S$ and $Q$ round $T$, we have

$$\frac{S+T+Q}{T+Q} \cdot \frac{TQ^3}{ST^3} = \frac{q^2}{p^2}.$$

(NEWTON, Book I. Prop. LX.)

58. PROP. *The force varying inversely as the square of the distance, it is required to compare the major axis of* P's *relative orbit round* S in motion, *with the major axis of the ellipse in which* P *might revolve round* S at rest, *in the same periodic time.*

Let $P'$ revolve round $S$ at rest, in the same time in which $P$ revolves round $S$ in motion; then by Prop. XV. Introd.

Major axis of $P'$'s orbit : major axis of $P$'s orbit

:: (Period)$^{\frac{2}{3}}$ of $P'$ : (Period)$^{\frac{2}{3}}$ of $P$ round $S$;

(by hyp.) :: (Period)$^{\frac{2}{3}}$ of $P$ round $C$ : (Period)$^{\frac{2}{3}}$ of $P$ round $S$;

(by last Prop.)

:: $S^{\frac{1}{3}}$ : $(S+P)^{\frac{1}{3}}$.

(NEWTON, Book I. Prop. LXI.)

59. PROP. *Two bodies attracting each other with any forces, it is required to find, for each of them, the body which must be placed in the centre of gravity, so that, attracting by the same law, it may produce the same effect.*

Let the force vary directly as the attracting body, and inversely as any power of the distance. Then the force of $S$ on $P$ is $\frac{S}{SP^n}$, $n$ being the index of the power. Let $X$ be the body which must be placed in the centre of gravity $C$ to produce the effect which $S$ produces in $P$. Then

$$\frac{X}{CP^n} = \frac{S}{SP^n}; \quad X = S \cdot \frac{CP^n}{SP^n} = S\left(\frac{CP}{SP}\right)^n = S\left(\frac{S}{S+P}\right)^n = \frac{S^{n+1}}{(S+P)^n}.$$

In like manner if $Y$ be the body which will produce on $S$ the effect which $P$ produces, $Y = \frac{P^{n+1}}{(S+P)^n}$.

COR. 1. If the force vary inversely as the square of the distance, the bodies will be respectively $\frac{S^3}{(S+P)^2}$, $\frac{P^3}{(S+P)^2}$.

COR. 2. If the force vary directly as the distance, the body to be placed in $C$ is $S + P$ for each of the bodies $S$ and $P$.

(NEWTON, Book I. PROP. LXIII.)

60. PROP. *Two bodies attract each other with forces which vary inversely as the square of the distance, and set out with given velocities in given directions: it is required to determine their motions.*

Fig. 153. Let $M$, $N$ be the original positions of the bodies, $MQ$, $NT$ their velocities, $B$ their centre of gravity. Let $TC : CQ :: NB : BM$; therefore $C$ is the centre of gravity of the bodies when they are at $T$, $Q$, and $BC$ is the velocity of the centre of gravity at first; and therefore always, because by Art. 55, the centre of gravity is either at rest, or moves in a straight line and uniformly.

Draw $PS$ parallel to $MN$; $MP$, $NS$ parallel to $BC$. Then $PQ$, $ST$ are the velocities of $P$ and $S$ about $C$; for the velocity $MQ$ is equivalent to $MP$, $PQ$; and the velocity $NT$

is equivalent to $NS$, $ST$; and of these, $MP$, $NS$ are the velocities arising from the motion of the centre of gravity, and the remaining parts $PQ$, $ST$ are the motions about the centre of gravity.

The body $S$ produces upon $P$ the same effect as a body $\frac{S^3}{(S+P)^2}$ placed at $C$. Hence if we suppose such a body fixed at $C$, and suppose the body $P$ to set out about this fixed body, with the velocity $PQ$, we may find the orbit described by $P$ about $C$, by Prop. XVII. of the Introduction. And in like manner we may proceed for $S$. And compounding the motion thus found with that of the centre of gravity $C$, we have the whole motion of $C$ and $S$ in absolute space.

## *Sect. II.* Problem of Three or more Bodies.

(Newton, Book I. Prop. LXIV.)

61. Prop. *If several bodies attract each other with forces varying as their masses and directly as the distance; it is required to determine their motions.*

Fig. 154. First, let there be two bodies $T$, $L$, of which $D$ is the common centre of gravity: by Art. 56, Cor. 1. these bodies will describe ellipses having their centre in $D$, and will move as if a body $T+L$ were placed in $D$, exerting a force $(T+L) \times$ distance. Art. 59. Cor. 2.

Next, let a body $S$ attract $T$ and $L$, and let $S$ and the line $TL$ have any relative motions. $S$ attracts $T$ and $L$ with forces which are as $S.ST$ and $S.SL$ respectively. The first of these is equivalent to $S.TD$ and $S.DS$, the second to $S.LD$ and $S.DS$. Therefore $T$ is acted on by forces $S.TD$ and $(T+L).TD$ tending to $D$, that is by a force $(S+T+L)$ $TD$ tending to $D$. In like manner $L$ is acted on by a force $(S+T+L)$ $LD$ tending to $D$. Also the equal and parallel forces $S.DS$ on $T$ and $S.DS$ on $L$ will not effect the relative motions of $T$ and $L$. Therefore $T$ and $L$ will revolve about $D$ as if a force $(S+T+L) \times$ distance resided in that point.

Let $C$ be the centre of gravity of $S$, $T$, $L$: then

$$S.SC = (T + L) DC,$$

and therefore $S.SD = S.SC + S.DC = (S + T + L) DC.$

Hence, the force $S.SD$ which acts on $T$ parallel to $DC$ is $(S + T + L) DC$: and this compounded with $(S + T + L) TD$ in the direction $TD$, gives a resultant $(S + T + L) TC$; and therefore $T$ is attracted towards $C$ as if there were at $C$ a force $(S + T + L) \times$ distance. In like manner $L$ and $S$ are attracted towards $C$ as if there were at that point a force $(S + T + L) \times$ distance.

Therefore $T$ and $L$ will describe about $D$, and $T$, $L$ and $S$ about $C$, ellipses having those points for their centres; and the periodic times in all these ellipses will be equal.

And the same might be proved if there were a greater number of bodies.

Cor. 1. In a system of any number of bodies, each describes about any other, and about the centre of gravity of itself and any others, ellipses about a centre; and the periodic times in all these ellipses are equal.

Cor. 2. In such a system, there will be no *Perturbations*; for all the bodies describe ellipses accurately. Also at the end of the periodic time the cycle of the changes of configuration is complete, and every one of the bodies returns to the position which it had at the beginning of the revolution.

62. In the preceding case the motions are accurately elliptical. But if the force do not vary directly as the distance, we cannot have accurate elliptical motion in a system of several bodies mutually attracting each other, except with certain peculiar relations of distance, &c. hereafter to be discussed. There are however some general cases in which the motions of several bodies, attracting each other with forces which vary inversely as the squares of the distances, will not much deviate from elliptical motion. (Newton, Book I. Prop. lxv.)

Case 1. Let several small bodies revolve about one much larger, at various distances; (as in the instance of the planets revolving about the sun.) Then the centre of gravity of the whole system will be very near the centre of gravity of the large body; and therefore the centre of the large body will either rest, or move uniformly in a straight line very nearly. The other bodies will revolve about the largest in ellipses or circles nearly, describing about it areas proportional to the times, except so far as errors occur, either from the deviation of the large body from the common centre of gravity, or from the mutual action of the bodies on each other. And by diminishing the smaller bodies, these errors may be diminished without limit.

Case 2. Let several small bodies revolve about a larger one, and let this system move transversely, acted upon by the attraction of a very much larger body, at a very great distance: (as in the instance of a planet with its satellites moving round the sun.) Then, since equal and parallel accelerating forces, acting upon the different points of a system, do not change the relative motions of the parts of the system, it is manifest that the action of the largest and distant body will not affect the relative motions of the smaller bodies, except in so far as it exercises unequal attractions upon the different bodies, or attractions in lines not parallel. Hence, if we suppose the largest body to be so distant that the differences, and the mutual inclination, of all lines drawn from it to different parts of the system of smaller bodies, may be neglected, the motions of this system will go on, with no errors, except such as may be neglected. The whole system will be attracted by the distant one, as if it were one body; and it will describe about the distant body an orbit determined by the velocity of the system and the attraction of the distant body, according to the preceding propositions.

In this case, the disturbance of the motions of the smaller bodies by the largest will be less, if they are all attracted equally at equal distances from the largest, than if, at equal distances, some are attracted more and others less.

For if one ($P$) be attracted more than the others, take away that part of the attraction which is common to that one ($P$) and to the others, and which (by the second Law of motion) does not disturb the relative motions. Then there remains a force which disturbs $P$'s relative motion, beside the disturbing forces arising out of the difference of distances of the different bodies from the largest body, and out of the mutual inclination of lines drawn to the distant body. Therefore the disturbance in this supposition is greater than on the other.

When the smaller bodies are all equally attracted, at equal distances, by the large and distant one, the deviations from regularity in their motions are small, and may be calculated. We shall first describe in a general manner, the nature of the perturbations thus occurring, and then explain how some of them may be calculated as to their law and quantity.

(NEWTON, Book I. Prop. LXVI.)

63. PROP. *To explain the principal perturbations in the motion of a system of three bodies attracting each other with forces varying inversely as the square of the distance.*

Fig. 155. Let a body $P$ revolve about a larger body $T$, by their mutual attraction; and let another body $S$ attract both $T$ and $P$.

First, let $STP$ be the plane of $P$'s motion.

Let $ST$ represent the accelerating force of $S$ on $T$; then the accelerating force of $S$ on $P$ at the same distance will also be represented by $ST$; and the accelerating force of $S$ on $P$ at $P$ will be $ST \cdot \frac{ST^2}{SP^2}$: take $SL$ equal to this. Then $SL$ represents the force of $S$ on $P$; and $LM$ being drawn parallel to $PT$, the force $SL$ may be resolved into forces $LM$, $MS$.

The body $P$ is acted on by three forces: 1st, the attraction of $T$ in the line $PT$, which varies inversely as the square of $PT$; 2d, the force $LM$, also in the direction $PT$, but varying according to some other law than the inverse square of the

distance; 3d, the force $MS$ which acts at $P$ parallel to $TS$. The force $MS$ is equivalent to $MT$, $TS$. If the two bodies $P$ and $T$ were acted upon by equal and parallel forces $TS$, $TS$, their relative motions would not be disturbed. Hence, the only forces which affect the relative motions of $P$ and $T$ are the attraction of $T$ on $P$, and the forces $LM$, $MT$. The former would cause $P$ to describe an ellipse about $T$ as a focus; the latter two forces tend to *disturb* this elliptical motion. The elliptical motion is disturbed on two accounts. The addition of the force $LM$ to the attraction of $T$ on $P$, causes the *law* of the force to differ from the inverse square of the distance; and the force $MT$, compared with the force in $PT$, causes the *direction* of the force to differ from the direction $PT$, which tends to the central body $T$.

Next, let the plane of the motion of $P$ round $T$ be inclined to the plane $STP$.

The forces may be represented and resolved as before. The force $LM$, which acts always in the line $PT$, will not disturb the plane of $P$'s motion. But the other force $MT$, acting at $P$, parallel to $TS$, will be inclined to the plane of $P$'s motion. Hence, the force $MT$ will tend to draw $P$ from the plane of its motion, and will therefore *disturb* the plane of the orbit.

If we use the language of Astronomy, the intersection of the plane of the orbit with a given plane passing through $S$ and $T$ (the *ecliptic*), is called the *line of nodes*. The motion of $P$, projected upon the given plane, is called the motion in *longitude*; the motion of $P$, perpendicular to the given plane, is called the motion in *latitude*.

When the body $P$ (referred to the given plane if necessary) is in the line $ST$, (as at $A$ or $B$), the body $P$ is said to be *in syzygy*, and the line $SATB$ is called *the line of syzygy*.

When the body $P$ (referred to the given plane if necessary) is in the line $CD$, perpendicular to $ST$, (as at $C$ or $D$), the body $P$ is said to be in *quadrature*, and $CD$ is called *the line of quadrature*.

It appears from what has been said, that the forces $LM$, $MT$, will both produce errors in the motion in longitude; and that the force $MT$, will produce errors in the motion in latitude, except the line of nodes coincide with the line of syzygy.

When $SP$ is less than $ST$, $SL$ will be greater than $SP$, and the force $MT$ will tend *towards* $S$. But when $SP$ is less than $ST$, as $Sp$, $Sl$ will be less than $Sp$, and the force $mT$ will tend *from* $S$.

Cor. 1. If the system $T$, $P$, revolve about a large and distant body $S$, (as in the case of the earth and moon revolving round the sun), the preceding reasonings are applicable to explain the perturbations of $P$'s elliptical motion round $T$.

We shall, in the following Corollaries, trace the principal inequalities which would affect the motion of $P$ in such a case, and shall point out the correspondence of these perturbations with the ascertained inequalities of the moon's motion.

### Variation.

64. Cor. 2. The force $MT$, during the motion of the body $P$ from $C$ to $A$, tends *in consequentia*, that is, to the side *towards* which the body is moving. Hence, it will cause the body to move further on that side than it would have done if no such force had acted. But if no such force had acted, the areas in equal times would have been equal; therefore, by the action of this force, the areas in succeeding times will always be greater than in the equal preceding times; and the description of areas is *accelerated* during the motion of the body $P$ from $C$ to $A$.

During the motion of the body from $A$ to $D$, the force $MT$ tends *in antecedentia*, or to the side *from* which the body $P$ is moving. Hence, it will cause the body to move less far on the

side towards which it is moving, than if there had been no such force; that is, the areas in succeeding times will always be smaller than in equal preceding times: the description of areas is retarded during the motion of the body *P* from *A* to *D*.

In like manner, it will appear that the description of areas is accelerated during the motion of *P* from *D* to *B*, and retarded during the motion from *B* to *C*.

Hence, the body *P* will describe areas quickest about the points of syzygy, *A*, *B*, and slowest about the points of quadrature *C*, *D*.

Cor. 3. If the orbit of *P*, independent of the disturbance, be a circle, the velocity in it is uniform: and hence, in order that the areas may be described quicker at *A* and *B* and slower at *C* and *D*, the velocity of *P* must be greater at *A* and *B* and smaller at *C* and *D*.

Cor. 4. The orbit of *P* (supposing the undisturbed orbit to be circular) will be more curved at the quadratures than at the syzygies. For the force being the same, the curvature is smaller as the velocity is greater. (Introd. Prop. IV.) And therefore on this account the curvature is least at syzygies and greatest at quadratures, by last Corollary. Also, the force at quadratures is increased by the force *LM*, (*MT* in this case vanishing or becoming very small;) and the force at syzygies is diminished by $MT - ML$, (*LM* in this case coinciding with *MT* in direction;) therefore, the force is greater at quadratures, and will produce a greater deflexion; and less at syzygies, and will produce a less deflexion. And on this account also, the curvature at syzygies is less than the curvature at quadratures.

Cor. 5. Hence, *P* will recede to a greater distance from *T* at quadratures than at syzygies; for the orbit must assume an oval form, the greatest curvature at *C* and *D*, being at the ends of the greatest diameter.

This is true only on the supposition that the undisturbed orbit is a circle.

If we calculate the motion of $P$ by the equations of motion, we obtain for the moon's parallax (which is the reciprocal of its distance) a series of terms depending on various angles. One of the principal of these terms is

$$m^2 \cos. \{(2 - 2m)\theta - 2\beta\}:$$

in which $\theta$ is the moon's longitude, and $m\theta + \beta$ the sun's. This term is greatest when $P$ is in syzygies, and least when $P$ is in quadratures, and thus the changes of the distance correspond with the form of the orbit described in Cor. 5. See *Airy's* Lunar Theory, Art. 62.

The error in longitude arising from the acceleration of areas (Cor. 2.) is as sin.2 (moon's mean longitude – sun's mean longitude) vanishing at quadratures and syzygies, and being the greatest in the *octants*, or points which are at the distance of $45^0$ from syzygy. The error in longitude arising from the oval form of the orbit (Cor. 5.) is also proportional to the same sine. Therefore the whole error in longitude arising from the disturbances described in the last four corollaries will be proportional to sin.2 (moon's mean longitude – sun's mean longitude). This error or inequality is, in the case of the moon, called the *Variation*. See *Airy's* Lunar Theory, Art. 64.

## ANNUAL EQUATION.

65. COR. 6. The force of the body $T$ on $P$, by which $P$ is retained in its orbit, is increased in quadratures by the addition of the force $LM$, and diminished in syzygy by the subtraction of the force $MT - ML$, and is on the whole more diminished than increased. And this total diminution of the mean central force is greater when the influence of $S$ is increased, as for instance, if $S$ approach nearer to $T$: and if $S$ alternately approach nearer to $T$ and recede farther from $T$, the total diminution of the mean central force will be alternately greater and less.

The mean central force of $T$ being diminished, the body $P$ will recede further from the centre $T$; and since $F$ is as $\frac{R}{P^2}$ (Introd. Prop. IV.) $P$ is as $\sqrt{\frac{R}{F}}$ and the periodic time $P$ will be increased both by the increase of $R$ and the diminution of $F$.

If the system $T$, $P$, revolve in an elliptical orbit about $S$, (as the earth and moon revolve about the sun,) $S$ being alternately at its least and greatest distances, (in perigee and apogee,) we have alternately $P$'s orbit dilated with a retarded motion of $P$, and $P$'s orbit contracted with an accelerated motion of $P$. Hence we have an error in $P$'s longitude produced, which vanishes when $S$ is in apogee and perigee, and is proportional to the sine of the distance from the perigee.

In the expression for the moon's longitude (*Airy*, L. T. Art. 64.) we have the term $-3me'\sin.(mpt+\beta-\zeta)$, where $e'$ is the eccentricity of the sun's apparent orbit, $mpt+\beta$ the sun's mean longitude, and $\zeta$ the longitude of the sun's perigee. This term corresponds to the error in longitude arising from the disturbance described in Cor. 6. and is called the *Annual Equation*.

### Progression of the Apsides.

66. Cor. 7. If $\frac{S}{ST^2}$ represents the force of $S$ on $T$, we shall have

$$\text{force } SL = \frac{S}{SP^2}, \text{ force } LM = \frac{S.PT}{SP^3}, \text{ force } MT = \frac{S.ST}{SP^3} - \frac{S}{ST^2}.$$

And when the body $S$ is very distant, compared with $TP$, $LM = PT$ very nearly, and $SP = ST - TK$, $PK$ being perpendicular to $ST$; hence

$$\text{force } MT = \frac{S.ST}{(ST-TK)^3} - \frac{S}{ST^2} = \frac{3S.TK}{ST^3}.$$

Let $\frac{S}{ST^3} = b$, $\frac{1}{TP^2}$ being the force of $T$ on $P$. Then at quadratures the force added to the attraction of $T$ on $P$ is as $PT$, and the whole force is as $\frac{1}{TP^2} + b \,.\, TP$. At syzygies the force subtracted is as $3AT - AT$ or as $2AT$, and the whole force is as

$$\frac{1}{TP^2} - 2b \,.\, TP.$$

Hence, by Art. 53, Cor. 1, in the neighbourhood of quadratures the apsides regress; but in the neighbourhood of syzygies, the apsides progress; and on the whole the progression exceeds the regression, because the force subtracted, $-2b \,.\, TP$ is double the force added, $b \,.\, TP$, and on the whole the apse proceeds in consequentia.

In the expression for the moon's parallax, the first and greatest inequality is $e \cos.(c\theta - a)$, (*Airy*, L. T. Art. 61) which shews that the moon moves in an elliptical orbit, of which the apse moves with a velocity which is to the moon's angular velocity as $1 - c : 1$.

The motion of the apse of the orbit of a satellite, so far as it depends on the part of the disturbing force which acts in the direction of the radius $PT$, may be calculated by finding the mean value of this disturbing force (which we shall do hereafter) and by applying the reasoning of Section 9, Chap. III. But by this method we do not take account of the effect of that part of the disturbing force which is not in the direction $PT$. In the case of the moon, this omission prevents the result calculated by Sect. 9. Chap. III, from being an approximation: it is only about half the true value. But in other cases, as in that of the satellites of Jupiter, the method of the 9th Section would be applicable as an approximation. The difference depends upon $m$, the ratio of the mean motion of the satellite ($P$) round the primary ($T$) to the mean motion of the primary ($T$) round the sun ($S$): $m$ for the moon is $\frac{1}{13}$; for Jupiter's outermost satellite it is $\frac{1}{260}$.

## EVECTION.

67. COR. 8. The progression of the apsides takes place in consequence of the deviation of the law of force from the inverse square of the distance; and the rate of progression is different with different forms and values of this deviation. Now the deviation of force from the law of the inverse square of the distance, produced by the disturbing forces, is different in the different positions of the line of the apsides. Thus, if $p$, $q$, $r$ be the least, greatest, and mean distances of $P$ from $T$, then, when the line of apsides is in syzygy, the forces at these distances will be respectively

$$\frac{1}{p^2} - 2bp, \quad \frac{1}{r^2} + bp, \quad \frac{1}{q^2} - 2bq:$$

but when the line of apsides is in quadratures, the three values of the force will be

$$\frac{1}{p^2} + bp, \quad \frac{1}{r^2} - 2bp, \quad \frac{1}{q^2} + bp:$$

and therefore in these different cases the motion of the apse will be different. In the former it will progress, and its motion will be more rapid than the mean; in the latter case it will regress, but the motion will be slower, the deviation from the law of the inverse square being in this case less than in the other. And this rapid progress of the apse will take place so long as the apse is in the neighbourhood of the syzygy, and the slower regress so long as the apse is in the neighbourhood of quadrature. And these periods will be considerable, for the apse is about a quarter of a year in moving from quadrature to syzygy. Hence this irregularity in the motion of the apse, so long continued, will give rise to a very considerable error in the place of $P$. The error thus produced is much the greatest of those arising from the disturbing forces.

Let $a$ be the longitude of the moon's perigee, supposing its motion not disturbed; $\beta$ the longitude of the sun. Then

it appears (*Airy*, L. T. Art. 66.) that the longitude of the moon's perigee, in order to account for the greatest inequality of her motion, must be $a - \frac{15m}{8}\sin.2(a-\beta)$. Hence if $\delta a$ and $\delta\beta$ be the increase of $a$ and $\beta$ in any time, the true motion of the perigee in that time will be

$$\delta a - \frac{15m}{4}(\delta a - \delta\beta)\cos.2(a-\beta).$$

Now in one revolution of the moon, the perigee advances a little more than $3^0$, or $\frac{1}{118}$ of the circumference nearly; in the same time the sun's longitude increases $\frac{1}{13}$ of a circumference nearly. Therefore, $\delta\beta : \delta a :: 118 : 13 :: 9 : 1$, nearly. And hence, the true motion of the perigee is

$$\delta a\left\{1 + 30m\cos.2(a-\beta)\right\},$$

When the perigee is in quadrature, $a-\beta$, or $\beta - a = \frac{\pi}{2}$, and the true motion is

$$\delta a(1-30m);\ \text{or, since } m = \frac{1}{13},\ \text{it is } -\delta a\frac{17}{13}:$$

and is therefore retrograde.

At the next octants $a - \beta = \frac{\pi}{4}$, and the motion of the perigee is $\delta a$, its mean motion.

At syzygy, $a - \beta = 0$,

$$\text{motion of the perigee} = \delta a(1+30m) = \delta a\frac{43}{13};$$

it is positive, and more than three times the mean motion.

The perigee is stationary for a moment

$$\text{when } 1 + 30m\cos.2(a-\beta) = 0,$$

$$\text{or } \cos. 2(\alpha - \beta) = -\frac{1}{30m} = -\frac{13}{30} = -,4333 \text{ \&c.,}$$

$$\text{whence } 2(\alpha - \beta) = 115^0\ 40, \quad \alpha - \beta = 57^0\ 50';$$

of which the complement is $32^0$ $10'$.

Hence when the perigee is in quadrature, it regresses, till, by the apparent motion of the sun, combined with its own motion, it is $32^0$ $10'$ from quadrature. It then begins to progress, and continues to do so through the syzygy, and till the sun is $57^0$ $50'$ beyond: after this the perigee again regresses till the sun is beyond the next quadrature: and so on.

The perigee is in its mean place at syzygy; it advances before the mean place more and more till the octant, after which it begins to diminish its speed: at quadrature it has fallen back to its mean place, and it falls behind this mean place more and more to the next octants; and so on.

Cor. 9. If, when a body is revolving in an ellipse, by means of a force varying inversely as the square of the distance, the force be made to vary in a higher inverse ratio than this, by the addition of some other force as the body approaches the centre, it is clear that the body will be drawn nearer to the centre than it would otherwise have been, and the excentricity will be increased. And if in the recess of the body from the centre, the force diminish in a more rapid proportion than that in which it had increased, the body will recede farther from the centre than it would otherwise have done, and the excentricity will be again increased. If therefore the ratio of the force at the lower apse to that at the higher apse go on increasing for several revolutions, the excentricity will during those revolutions perpetually increase: and *vice versâ*, if that ratio diminish, the excentricity will diminish.

Now the ratio of the force at the lower to that at the higher apside, if $p$ and $q$ be the least and greatest apsidal distances, is

with the apsides in quadratures

$$\frac{\frac{1}{p^2} + bp}{\frac{1}{q^2} + bq};$$

with the apsides in syzygies

$$\frac{\frac{1}{p^2} - 2bp}{\frac{1}{q^2} - 2bq};$$

$$\text{or,}\quad \frac{q^2}{p^2}\,\frac{1 + bp^3}{1 + bq^3}, \quad\text{and}\quad \frac{q^2}{p^2}\,\frac{1 - 2bp^3}{1 - 2bq^3},$$

or since $b$ is small,

$$\frac{q^2}{p^2}\left\{1 - b(q^3 - p^3)\right\} \quad\text{and}\quad \frac{q^2}{p^2}\left\{1 + 2b(q^3 - p^3)\right\},$$

nearly; of which the latter is the greater; and hence, the ratio goes on increasing from the former to the latter period. Therefore the excentricity goes on increasing from the time of the apsides being in quadratures, to the time of their being in syzygies; and *vice versâ*, the excentricity diminishes from the syzygy to the quadrature of the apsides.

Let, as before, $\alpha$ be the longitude of the moon's perigee, $\beta$ the longitude of the sun, $e$ the mean excentricity of the moon's orbit; then the true excentricity will be

$$e\left\{1 + \frac{15}{8} m \cos.2(\alpha - \beta)\right\};\ \textit{Airy},\ \text{L. T. Art. 66.}$$

Hence when the perigee is in quadrature or $\alpha - \beta = \frac{\pi}{2}$, the excentricity $= e\left\{1 - \frac{15}{8} m\right\}$ its least value; when the perigee comes into octants, $\alpha - \beta = \frac{\pi}{4}$, and the excentricity $= e$ its

mean value: at syzygy the excentricity $= e\left\{1 + \frac{15}{8}m\right\}$ which is the greatest value. And this agrees with the corollary.

*Construction*, (NEWTON, Vol. III. p. 528.)

Fig. 156. Let $TC$ represent $e$ the mean excentricity of the moon's orbit, and $CB = CA = \frac{15me}{8}$;

and let $BCD = 2(a - \beta) = 2$ distance of mean perigee from sun.

Then $TD$ nearly $= TE = TC - CE = e\left\{1 + \frac{15m}{8}\cos. 2(a - \beta)\right\}$;

$$\text{and angle } CTD = \frac{DE}{TD}, \text{ nearly, } = \frac{DE}{TC}, \text{ nearly}$$

$$= \frac{15m}{8}\sin. 2(a - \beta).$$

Hence if $T$ be the place of the earth, and $TC$ the direction of the mean position of the moon's apogee, $TD$ will be direction of the apogee as effected by evection. The centre of the moon's orbit revolves in the *epicycle* $BDA$.

The perturbation of the place of the apside and the alteration of the value of the excentricity will both affect the *elliptical inequality* or equation of the centre; and the error of longitude arising from these two causes may be expressed by a single term. For the equation of the centre, instead of $e\sin. (\theta - a)$ becomes, altering $e$ and $a$ as above,

$$e\left\{1 + \frac{15}{8}m\cos. 2(a - \beta)\right\}\sin. \left\{\theta - a + \frac{15}{8}m\sin. 2(a - \beta)\right\}.$$

Expanding, and neglecting terms involving $m^2$, we have

$$\sin. \left\{(\theta - a) + \frac{15}{8}m\sin. 2(a - \beta)\right\}$$

$$= \sin. (\theta - a) + \frac{15}{8}m\cos. (\theta - a)\sin. 2(a - \beta);$$

Whence the equation of the centre becomes

$$e \sin. (\theta - a) + \frac{15}{8} me \sin. \{(\theta - a) + 2 (a - \beta)\},$$

the last term is the *inequality* now considered.

There is in the expression for the moon's longitude, a term (*Airy*, L. T. 59,)

$$\frac{15}{8} me \sin. \{(2 - 2m - c)pt - 2\beta + a\}.$$

Now, $c$ is not much different from 1, and $m$ is small; whence the arc in this term is nearly

$$pt - 2\beta + a, \text{ or nearly } \theta - 2\beta + a,$$

which is the same as the arc in the above expression for the inequality.

The angular distance of the sun and moon is $\theta - m\theta - \beta$, and the moon's anomaly is $c\theta - a$; therefore the arc

$$2\theta - 2m\theta - 2\beta - c\theta + a,$$

is twice the distance of the sun and moon *minus* the moon's anomaly. And the inequality above described is proportional to the cosine of this arc. This inequality is called the *Evection.*

### Change of the Inclination.

68. Cor. 10. Suppose $P$'s orbit to be inclined to a given fixed plane passing through $S$. Then, the forces $LM$, $MT$ act as before; and of these, $LM$, which always acts in the line $PT$, will not disturb the plane of $P$'s orbit. But the force $MT$, acting at $P$ parallel to $TS$, will urge the body from the plane of its orbit, except when the line of nodes coincides with $TS$ and $STP$ is the plane of the orbit.

When the line of nodes is in quadrature, let the body $P$ set out from quadrature towards syzygy: the force $MT$ will at

every point act to urge the body *P* towards the fixed plane, and as the body is moving from the fixed plane, the force will manifestly at every instant diminish the inclination. When the body has passed the syzygy, and is moving to the next quadrature, it is moving towards the fixed plane; and as the force still acts towards the plane, the inclination of the motion to the plane will be constantly increased, till the body reaches the node. Hence, in this position of the line of nodes, the inclination of the orbit will be diminished as the body moves from quadratures, will be least when the body is in syzygies, and will return at the next quadrature nearly to its original magnitude.

When the line of nodes is in octants after quadratures, that is, between *C*, and *A*, or *D* and *B*, it will appear, by similar reasoning, that while the body *P* moves from any node to the point $90^0$ from it, the inclination will be constantly diminished; for the next $45^0$ till the body reaches the following quadrature, the inclination will be increased; for $45^0$ more, from quadrature to the node, it will be diminished. Thus, it is diminished through $135^0$ and increased through $45^0$, and is therefore more diminished than increased in a half revolution; and the same effect precisely will be produced in the next half revolution. It is therefore perpetually less and less at every succeeding appulse of the body to the node. And by a similar reasoning it will appear that the same is true, so long as the nodes are between *A* and *C*, and *B* and *D*.

When the nodes are in the octants before quadratures, it may be proved, in like manner, that the inclination will be increased through $135^0$ of *P*'s semi-revolution, and diminished through $45^0$; and therefore it will be, on the whole, more increased than diminished; and the inclination will be greater and greater at each appulse of the body to the node. And the same is true, so long as the nodes are between *B* and *C*, and *A* and *D*.

Therefore, while the node moves from *A* to *C*, the inclination of the orbit is perpetually diminished. It is least when the line of nodes is in quadrature. After that point it increases perpetually while the node moves from *C* to *B*, and

is greatest when the node is in syzygy, increasing by the same degrees as those by which it had diminished, and returning to its original magnitude.

In the case of the moon, we find (*Airy*, L. T. Art. 68,) that if $\gamma$ be the longitude of the node, $\beta$ the longitude of the sun, the tangent of the inclination of the orbit is

$$k\left\{1 + \frac{3m}{8}\cos. 2(\gamma - \beta)\right\};$$

which is greatest when $\gamma - \beta = 0$ or $= 180^0$, that is, when the nodes are in syzygy; and least when $\gamma - \beta = 90^0$ or $= 270^0$, that is, when the nodes are in quadrature.

### Regression of the Nodes.

69. Cor. 11. When the nodes are in quadratures, the body $P$ is perpetually drawn from the plane of its orbit towards the fixed plane $EST$, by the force $MT$; and in moving from $C$ to $D$, is urged from the plane of its orbit towards the side on which is $S$. Hence, it will recede from the plane of its orbit, and will meet the fixed plane $EST$, not at the node $D$, but at a point situate from $D$ towards $S$, which is the new place of the node. And, in like manner, during the motion of the body from the node $D$ to $C$, the node will be shifted from $C$ to the side opposite to $S$. Therefore, when the nodes are in quadrature, the line of nodes regresses during a revolution of the body $P$.

When the nodes are in syzygy, they remain stationary during a revolution of the body $P$.

When the nodes are in any intermediate position, they will have an intermediate motion; that is, they will regress more slowly: (as will hereafter be more fully shewn).

Hence, upon the whole, the nodes regress.

In the expression for $s$, the tangent of the latitude of the moon, we find (*Airy*, L. T. 67,) a term $k$ sin. $(g\theta - \gamma)$, which

represents motion in a plane, the tangent of whose inclination to the ecliptic is $k$, the nodes of the plane being supposed to move with a retrograde motion, which is to the whole motion of the moon as $g-1 : 1$.

### Difference of Magnitude of Errors.

70. Cor. 12. If $\frac{S}{ST^2}$ represent the force of $S$ on $T$, we shall have, as in Cor. 7,

$$\text{force } SL = \frac{S}{SP^2}, \quad \text{force } LM = \frac{S.LM}{SP^3},$$

$$\text{force } MT = \frac{S.ST}{SP^3} - \frac{S}{ST^2}:$$

and these forces are somewhat greater when $P$ is at $A$ than when $P$ is at $B$, $SP$ being smaller in the former case; therefore all the preceding errors are somewhat greater in the *conjunction* of $P$ and $S$ (as seen from $T$), than in their opposition.

Cor. 13. The preceding reasonings apply both to cases where a system $T$, $P$, revolves round a larger $S$, (as the earth and moon round the sun); and to those where different bodies $P$, $S$, revolve round a central body $T$, (as the planets round the sun). But the body $S$ being much larger in the former case than in the latter, the errors are also, at equal distances, much larger in that case.

### Comparative Errors in different Systems.

71. Cor. 14. When the body $S$ is very distant, we have, as in Cor. 7,

$$\text{force } LM = \frac{S.PT}{ST^3}, \quad \text{force } MT = \frac{3S.TK}{ST^3}.$$

And when the force and distance of $S$ vary, the system $T$, $P$ remaining, these forces are as $\frac{S}{ST^3}$.

But when a body or system revolves in a circle about a centre of force $S$, at a distance $ST$, we have force $\propto \frac{ST}{P^2}$, $P$ being the periodic time round $S$; or

$$\frac{S}{ST^2} \propto \frac{ST}{P^2}; \quad \text{whence } \frac{S}{ST^3} \propto \frac{1}{P^2}.$$

(*Introd.* Prop. IV.). Hence, the disturbing forces are inversely as the square of the periodic time.

If the absolute force $S$, at a given distance, be proportional to the magnitude of the body $S$, let $D$ be the diameter of $S$: then $S \propto D^3$, and the disturbing forces are as $\frac{D^3}{ST^3}$, and therefore as the cube of the *apparent* diameter of $S$.

Cor. 15. If the form, proportion, and inclination of the orbits $ESE$ and $PAB$, and the relative forces of $S$ and $T$, remain unchanged; and if the magnitude of the orbits be altered in any ratio, the *relative* directions and inclinations of the forces at every instant will be the same as before; the motions will be similar, and the times of similar motions will be proportional.

That is, the linear errors in this case will be altered proportionally to the diameters of the orbits; the angular errors will be the same as before; and the times of similar linear, or equal angular errors, will be altered proportionally to the periodic times of the orbits.

Cor. 16. Having given the forms and inclination of the orbits, and, knowing the errors and their times in one case, to find the errors and their times, when the magnitudes, forces and distances of the bodies are changed in any manner.

When $TP$ is altered, other things remaining, the forces $LM$, $MT$ are altered proportionally to $TP$: and the spaces through which they similarly draw the bodies will be as the forces × the squares of the times, (*Introd.* Lemma x.); therefore, the linear errors in the course of $P$'s period, (which are

spaces similarly described), will be as $TP \times$ the square of $P$'s periodic time.

Therefore, in one period of $P$, the angular errors as seen from $T$ are as the square of $P$'s periodic time.

Hence, the motion of the apse and of the nodes in the course of one revolution of $P$, and all the apparent errors in latitude and longitude, will be in this proportion.

This is also true, if $T$ be altered as well as $TP$.

By Cor. 14, when the force and distance of $S$ vary, the errors in the system $T$, $P$, vary inversely as the square of the periodic time of $T$ and $P$, round $S$. Hence, if both $S$, $ST$ and $T$, $TP$ vary, the angular errors are as squares of the periodic time of $P$ round $T$ directly, and the square of the periodic time of $T$ round $S$ inversely.

Hence, for different satellites (the excentricity and inclination being the same), the mean motion of the apse and the mean motion of the node in the course of one revolution, are each as the square of the period of the satellite directly, and the square of the period of the primary inversely.

And for different satellites of the same primary (the excentricity and inclination being the same), the motion of the nodes is in a given ratio to the motion of the apse.

The motion of the nodes and of the apse are not sensibly changed by altering the excentricity and inclination, except these be considerable.

It appears by calculation, that if $m$ be the ratio of the periodic time of the satellite to that of the primary, $1 - c$ the motion of the apse, and $g - 1$ the motion of the node, as compared with that of the satellite, we have (*Airy*, L. T. Art. 72,)

$$c = 1 - \frac{3}{4}m^2 - \frac{225}{32}m^3 + \&c.$$

$$g = 1 + \frac{3}{4}m^2 - \frac{9}{32}m^3 + \&c.$$

Hence, it appears that if we neglect $m^3$, &c. as small, the progression of the apse in one revolution of the satellite is equal to the regression of the node, each being $\frac{3m^2}{2}\pi$.

The corollary just proved cannot be strictly applied to the moon; for in the case of the moon $m = \frac{1}{13}$, therefore

$$\frac{225}{32}m^3 = \frac{225}{32 \times 13}m^2 = \frac{7}{13}m^2, \text{ nearly,}$$

which is not much less than the preceding term $\frac{3}{4}m^2$.

Hence, in applying this corollary, in order to obtain, from the motion of the apse of the moon, that of a satellite for which $m$ is small, the motion thus found would be too great, nearly in the ratio of $\frac{3}{4} + \frac{7}{13}$ to $\frac{3}{4}$; which is not much different from the ratio of 9 to 5. Newton, when in Book III. Prop. 23, he applies this corollary to Jupiter's satellites, reduces the amount in the ratio of 9 to 5, but does not explain the reason.

In the case of Jupiter's 4th or outermost satellite, $m$ is about $\frac{1}{264}$. In the case of Saturn's 7th satellite, $m$ is about $\frac{1}{185}$. In these cases the corollary might be applied without sensible error, and *a fortiori* for the inner satellites.

Cor. 17. If $F$ represent the force which $T$ exerts upon $P$ to retain in its orbit, $P$ the period of $T$ round $S$, $p$ the period of $P$ round $T$, since force $\propto \frac{\text{rad.}}{\text{per.}^2}$ in circles,

$$F : \frac{S}{ST^2} :: \frac{TP}{p^2} : \frac{ST}{P^2}; \text{ whence } \frac{S \cdot TP}{ST^3} = F\frac{p^2}{P^2}.$$

Hence, by Cor. 14,

$$\text{force } LM = F\frac{p^2}{P^2}; \quad \text{force } MT = F\frac{p^2}{P^2}\cdot\frac{3\,TK}{TP}.$$

### Motion of a Fluid Ring.

72. Cor. 18. Suppose a number of bodies $P$ to revolve round $T$, at the same distance from it: suppose these to become so numerous that they touch each other and form a ring; and suppose the bodies to be placed so that we may have a fluid ring surrounding the body $T$. The parts of this ring will move in all respects like the body $P$ in the preceding corollaries, and thus they will move quicker in their conjunction and opposition with $S$, and slower at quadratures, (Cor. 3.). Also the nodes of this ring, that is its intersections with the fixed plane $EST$ will be at rest when they are in syzygies, but in other positions they will regress, quickest at quadratures, slower in other places, (Cor. 11.). Also the inclination of the ring will change (Cor. 10,) and its axis will oscillate in the course of each revolution; and at the end of the revolution will return to its former position, except in so far as it is affected by the regression of the nodes.

### Tides.

73. Cor. 19. Now, suppose the globular body $T$ to be solid, and to be extended in dimensions till it meets this fluid ring, and to contain the fluid in a canal which runs round the globe. This fluid will then be alternately accelerated and retarded as in the last corollary; at the syzygies it will move quickest, at the quadratures slowest, and thus, relatively to a point moving with the mean velocity, there will be a flux and reflux, resembling the tides. Also the water will be highest in the part of the canal when it moves slowest, that is, at quadrature; and lowest in the part where it moves quickest, that is, at syzygy.

If the fluid were to revolve about the centre of a globe at rest, not acted upon by a body $S$, there would be no flux and

reflux. The same is true of a globe which moves uniformly forwards, and at the same time revolves about its centre; and also of a globe which is made to deviate from its rectilinear course by any force. (See the 2d Law of motion.)

But if the body $S$ now act upon the globe; the fluid will be disturbed by its unequal action. For the nearer fluid will be more attracted, the more remote fluid less attracted than the centre; and hence, there will be disturbing forces $LM$, $MT$ as in the proposition. The force $LM$ urges the fluid towards the centre equally at all points of the circumference, and produces no tide. The force $MT$ may be resolved into two forces, one in the direction of the radius, the other perpendicular to this direction, and acting towards the syzygy. In consequence of the action of this latter force, the fluid will be highest at quadrature and lowest at syzygy, except so far as the motion of flux and reflux is diverted by the form of the canal or retarded by friction.

And if the centripetal force of the globe increase, so that all the parts tend to its centre, after the manner of bodies gravitating on the Earth's surface, the phenomena already stated in this corollary will be scarcely altered; except that the places of the greatest and least altitudes of the fluid will be different. For in this case the water is kept in its permanent round form, not by its centrifugal force, as in Cor. 18, but by its gravity, which retains it in the channel in which it flows. And the part of the disturbing force which is in the direction of the radius will cause its gravity to be different in different parts of the circumference. For the force $LM$ draws it downward at quadratures, and the force $MT - LM$ draws it upward at syzygies. And in consequence of these forces the fluid would be lowest at quadrature and highest at syzygy.

The joint effect of the disturbing forces will attain its maximum somewhere after syzygy and before quadrature. Hence, the greatest altitude of the water may be at octants after syzygies, and the least at octants after quadratures

nearly; except so far as the motion of ascending and descending produced by these forces continues longer in consequence of the inertia of the water, or is stopped sooner in consequence of the resistance offered by the channel.

### Precession of the Equinoxes.

74. Cor. 20. Let the fluid ring in Cor. 18, now become solid: the motion of flux and reflux will now cease to take place, but the oscillatory change of inclination, and the precession of the nodes will still exist. Let the globe *T* fill the interior of the rigid ring, and revolve in the same time with the ring, adhering to it, so as to form one mass. Then the whole mass consisting of globe and ring, will share in such motions of the ring as have been described; its axis will oscillate and its nodes will regress. For the globe is, by its inertia, equally fitted to receive all impressed motions, and to retain them when received.

When the ring is not attached to a globe, its greatest angle of inclination is when the nodes are in syzygy (Cor. 10.). In the passage of the nodes from syzygies to quadrature, the ring *tends* to diminish its inclination, and by this tendency impresses its motion in the whole globe. The globe retains the motion impressed till the ring by its inverted tendency has destroyed it; after which the ring impresses upon the globe a new motion in the opposite direction.

By this means the most rapid diminution of the inclination takes place when the nodes are in quadratures, but the least angle of inclination in the octants after quadratures; the most rapid increase of inclination at syzygies, the greatest inclination in the succeeding octants.

The same things would happen with regard to a globe with no ring, if it were either more protuberant towards the equator than at the poles, or consisted in the former part of denser matter. For the excess of matter in the equatorial regions answers the purpose of the ring before supposed.

Cor 21. In the same manner in which an excess of matter in the neighbourhood of the equator causes the nodes to regress, and produces a greater regression the greater is the excess, a less regression when the excess is less, and no regression when the excess disappears; if we take away more than the redundant matter, that is, if there be a depression of the globe at the equator, or if its material be there less dense, the nodes of the globe will progress.

Cor. 22. Hence, conversely, by knowing the direction of the motion of the nodes we know the constitution of the globe. If the globe always keeps the same poles, and its nodes move backwards, there is an excess of matter at the equator: if the nodes move forwards, there is a defect of matter there.

## Composition of Rotatory Motion.

75. Let a perfectly spherical body be at rest in free space; and afterwards let it receive some oblique impact, so as to acquire a motion both of translation and of rotation. It will afterwards, by its inertia, retain the same axis in the same position, without any further change. Let the globe again receive a new oblique impact on the same point of the surface as before: and as it can make no difference whether the impulse take place soon or late, the two impulses will produce the same effect as if they had taken place at the same time; that is, the same effect, as if a single impulse compounded of the two had taken place. The globe will therefore have a simple rotatory motion, about a given axis. The effect will be the same if the second impulse takes place at any point of the *equator* of the first motion; and also if the first impulse take place at any point of the equator of the second motion. Hence the two impulses will generate a circular motion which will be the same as if both impulses had acted at the *intersection of the two equators:* and this will be true if the two impulses act at any points whatever. Therefore a homogeneous sphere does not retain separate rotatory motions, but compounds them all into one uniform motion about an invariable axis in a constant position.

The centrifugal force of a revolving globe cannot change the inclination of the axis or the velocity of rotation. But if there be added to any part of the globe between the pole and the equator a new portion of matter, as a mountain, this, by its perpetual tendency to recede from the axis of motion, will disturb the motion, and cause the poles of rotation to shift along the surface of the globe, describing circles about the mountain and the point opposite to it. And this irregular wandering of the poles will not be corrected, except when the mountain is at the pole, in which case (by Cor. 21,) the nodes of the equator will progress; or at the equator, in which case the nodes regress; or by adding another portion of matter on the other side of the equator, (on the same side of the axis;) and in this case the nodes either advance or recede, as the excess of matter is nearer to the pole or to the equator.

If an oblate spheroid be conceived to be at rest and moveable about its centre, and be attracted by a distant body *S*, the attraction would impress upon the spheroid a rotatory motion round a certain axis. And if the spheroid have previously a rotatory motion about its axis of form, these two rotatory motions will be compounded, and will produce rotation about a new axis. And if the attraction of *S* act perpetually, the axis of rotation of the spheroid will be perpetually altered. But this alteration of the instantaneous axis of rotation will be such that the motion of the spheroid may be considered as compounded of a rotatory motion round its axis of form, and a regression of the nodes of its equator upon the plane *ESE*, such as is described in Cor. 21.

By calculating the effect of the sun's action upon the terrestrial spheroid, on the above principles, it is found that there is produced a precession of the equinoxes, of which the amount in one year is (*Airy*, Precession, Art. 44,)

$$\frac{B \,.\, 3\pi \,.\, \cos. 23^{\circ}\, 28'}{366{,}26},$$

where $B$ is a quantity depending upon the law of the earth's density.

## *Sect. III.* Calculation of some of the Lunar Inequalities according to Newton's Method.

(Principia, Book III. Prop. xxv—xxxv.)

(Newton, Book III. Prop. xxv.)

76. Prop. *To find the force of the sun to disturb the motion of the moon.*

Fig. 157. Let $S$ now represent the sun, $T$ the earth, $P$ the moon, $G$ the centre of gravity of the earth and moon. By Art. 55. $T$ and $P$ by their mutual action revolve about $G$; and each of these bodies is disturbed in its motion by forces similar to the forces $LM$, $MT$ which are considered in Prop. 66 and its corollaries, (Art. 63 to 74.) Hence it appears by Cor. 7. (Art. 66.) that if $F$ be the force by which $P$ is retained in its orbit, $p$ the period of $P$ or $T$ round $G$, $P$ the period of $T$ round $S$, the disturbing forces which act upon $P$ are $F\frac{p^2}{P^2}$ in the direction $PT$, and $F\frac{p^2}{P^2}\frac{3GK}{GP}$ in a direction parallel to $GS$. Similar forces act upon $T$; and if $G$ be the force which retains $T$ in its orbit, the disturbing forces upon it are respectively, $G\frac{p^2}{P^2}$ and $G\frac{p^2}{P^2}\frac{3GH}{GT}$, or $G\frac{p^2}{P^2}\cdot\frac{3GK}{GP}$.

The disturbance of $P$'s orbit *relatively* to $T$ will be the sum of these two sets of disturbances; the *relative* disturbing force, in the direction of the radius $PT$, for instance, will be $(F+G)\frac{p^2}{P^2}$. But the *relative* motion of $P$ towards $T$ is the sum of the motions of $P$ and $T$ towards $G$; and hence the effects of this kind which the mutual forces of $P$ and $T$ produce will be as $F+G$. Therefore the relative disturbance of the motion of $P$ has to the relative motion of $P$ round $T$ at rest in the period $p$, the same relation which we found for the absolute disturbance of the absolute motion in Prop. 66.

If $F'$ be the force with which $P$ would revolve round $T$ at rest at the distance $TP$ in the time $p$, the relative disturbing force on $P$ in $PT$ is $F'\dfrac{p^2}{P^2}$.

The moon $P$ and the earth $T$ revolve round $G$ in the same time $p$, their distance being $60\frac{1}{3}$ radii of the earth. And if $P$ were to revolve round $T$ at rest in the same time $p$, their distance would be less in the ratio $\sqrt[3]{T+P} : \sqrt[3]{T}$. (Art. 58.) Supposing $T = 60P$, this ratio is $\sqrt[3]{61} : \sqrt[3]{60}$ or $60\frac{1}{3} : 60$ nearly; whence it follows that the distance would on this supposition be 60 radii. If $F''$ be the force which would retain $P$ in its orbit at this distance,

we have $F' : F'' :: 60\frac{1}{3} : 60$ and $F' = F''\dfrac{60\frac{1}{3}}{60}$.

The force by which $P$ is retained in its orbit about $T$ at rest is the attraction of the earth, that is, the force of gravity diminished in the ratio of the inverse square of the distance. If $g$ be gravity at the earth's surface,

$$F'' = g \cdot \frac{1}{60^2}; \quad F' = g\frac{60\frac{1}{3}}{60^3}.$$

Hence we have for the relative disturbing forces,

$$\text{force in the direction } PT = F'\frac{p^2}{P^2} = g\frac{60\frac{1}{3}}{60^3}\frac{p^2}{P^2},$$

$$\text{and force parallel to } GS = g\frac{60\frac{1}{3}}{60^3}\frac{p^2}{P^2}\cdot\frac{3GK}{GP}.$$

Also,

$$\frac{p}{P} = \frac{27,322}{365,25} = \frac{1}{13,4} \text{ nearly; and } \frac{p^2}{P^2} = \frac{1}{178,725} \text{ nearly.}$$

Hence we have the proportions of the disturbing forces to the force of gravity.

77. Let the angle $PGS$ be called $\omega$; then, retaining the former notation, the forces are $F'\dfrac{p^2}{P^2}$ in direction $PT$, $3F'\dfrac{p^2}{P^2}\cos.\omega$ parallel to $GS$.

The latter force may be resolved into two portions, in the direction $TP$, and perpendicular to that line, respectively. And if $PR$, $RQ$ be these portions, it will easily be seen that they are respectively

$$3F'\frac{p^2}{P^2}\cos.\omega\cos.\omega, \quad \text{and} \quad 3F'\frac{p^2}{P^2}\cos.\omega\sin.\omega.$$

Hence the whole *radial* disturbing force (in the direction of the radius $PT$) is

$$F'\frac{p^2}{P^2}\{1-3\cos.^2\omega\} = F'\frac{p^2}{P^2}\left\{-\frac{1}{2}-\frac{3}{2}(2\cos.^2\omega-1)\right\}$$

$$= -F'\frac{p^2}{P^2}\frac{1+3\cos.2\omega}{2}.$$

And the *transverse* disturbing force (in the direction perpendicular to the radius $PT$) is

$$3F'\frac{p^2}{P^2}\cos.\omega\sin.\omega = F'\frac{p^2}{P^2}\frac{3\sin.2\omega}{2}.$$

The *radial* force $-F'\frac{p^2}{P^2}\frac{1+3\cos.2\omega}{2}$, varies as $\omega$ varies; and $\cos.2\omega$ in the course of half a revolution of $P$ goes through all its values, having negative values equal in number and magnitude to the positive ones. Hence the *mean* value of the radial force will be had by neglecting the variable portion depending on $\cos.2\omega$, which thus destroys its own effect; and the mean value is therefore, $-\frac{F'}{2}\frac{p^2}{P^2}$. It tends from the centre.

Since $\frac{p^2}{P^2}=\frac{1}{178{,}725}$ the mean radial disturbing force is $-\frac{F'}{357{,}45}$, which is that used in page 95, in calculating the motion of the apse.

The amount of the motion of the apse thus found, requires correcting in the case of the moon, in consequence of the *transverse* disturbing force which we have also found.

78. PROP. *The transverse disturbing force cannot affect the motion of the apse, except when the square of the disturbing force is taken into account.*

The radial disturbing force alters the direction of the body's motion, whether it be at an apse or not; and thus will, by its action in a given small element of the orbit, cause the curve not to be perpendicular to the radius when otherwise it would have been so, and to be perpendicular when otherwise it would not have been so; and thus it may cause the place of the apse to change. But the transverse disturbing force, being perpendicular to the radius, cannot by its action in a given small element, make the curve cease to be oblique to the radius if it be so, or cease to be perpendicular if it be perpendicular to the radius; that is, this force cannot make an apse where without the force it would not have been, nor remove it where it is. And when we suppose the disturbing forces very small, so that we may neglect their squares, the disturbance produced in one element of the curve will not affect the disturbance produced in the other elements. Hence, on this supposition, the apse will not be removed by the action of the transverse disturbing force.

But if the disturbing forces be larger, the transverse force will, by acting during a finite time, produce an error in the place of the body, and the body's position will not be such as it would have been without the action of the disturbing force. And the radial disturbing force depends for its value upon the position of the body, and hence will be different from what it would have been if the transverse force had not acted. Therefore the motion of the apse will thus be affected mediately, though not immediately, by the action of the transverse force.

The error of the moon's place due to the transverse force, and the error in the quantities on which the radial force depends, are, *ceteris paribus*, as the disturbing force. And the effect of such an error is as the portion of the disturbing force which arises from the error, that is, as the disturbance due to a disturbance, or as the square of the disturbing forces.

In the case of the outermost satellites of Jupiter and Saturn, the disturbing force has the factors

$$\frac{1}{(264)^2}, \text{ or } \frac{1}{(135)^2};$$

and the other satellites of these systems have still smaller multipliers. In these cases the effect of the transverse force may be neglected. But in the case of the moon $\frac{p^2}{P^2} = \frac{1}{178{,}725}$; and the next power cannot safely be neglected.

The true motion of the perigee of the moon's orbit compared with her mean motion is

$$\frac{3}{4}\frac{p^2}{P^2} + \frac{225}{32}\frac{p^3}{P^3} = \left\{\frac{3}{4} + \frac{225}{32 \times 13{,}4}\right\}\frac{p^2}{P^2}. \quad (\textit{Airy}, \text{ L. T.})$$

If we neglect the second term we have $\frac{3}{4}\frac{p^2}{P^2}$, which agrees with Art. 53; but it is manifest that in this case the second term is nearly as large as the first. Including both in the calculation, we find for the motion of the apse in one revolution, $3^0{,}03588$.

(Newton, Book III. Prop. xxvi.)

79. Prop. *To find the horary (or instantaneous) increment of the area which the moon describes about the earth.*

This increment is produced by the transverse disturbing force, (Art. 65. Cor. 2.) and this force is, for the relative motion $3(F + G)\frac{p^2}{P^2}\sin.\omega\cos.\omega$. (Art. 76.) where $\omega$ is $PGS$.

Fig. 157. Let the orbit of the moon be supposed to be circular, and in the plane of the ecliptic. Let $EG$ be the origin of longitudes; let $EGP$ be the longitude of the moon seen from $G = \theta$, $EGS$ the longitude of the sun also seen from $G = \theta'$. If the motion of the moon and the apparent motion of the sun be circular, their angular velocities will have a given ratio,

and the increase of their longitudes in the same time will always be in this ratio; it will be the ratio of their periodic times, because these are the times of describing equal arcs, namely a whole circumference. Let $m$ be the ratio $\frac{p}{P}$; and let $\beta$ be the longitude of $S$ when $\theta = 0$,

then $\theta' = m\theta + \beta$, and $\omega = \theta' - \theta = m\theta - \theta + \beta$.

Hence if $\delta$ designate the small increment, $\delta\omega = -(1-m)\delta\theta$.

Now if $r$ be the radius of $P$'s relative orbit, $V$ the velocity in this orbit, $V^2 = (F+G)r$ (by the property of circular motion), also $\delta t = \frac{r\delta\theta}{V} = -\frac{r\delta\omega}{(1-m)V}$.

The area in time 1, is $\frac{1}{2}rV$. The increment of this area arising from the transverse force, depends on the increment of the velocity $V$, and we have

$$\delta V = \text{force} \times \delta t$$

$$= 3(F+G)m^2 \sin.\omega\cos.\omega\,.\,\delta t = -\frac{3(F+G)rm^2}{(1-m)V}\sin.\omega\cos.\omega\,.\,\delta\omega$$

$$= -\frac{3Vm^2}{(1-m)}\sin.\omega\cos.\omega\,.\,\delta\omega.$$

Hence, integrating so as to find the augmentation of velocity, we find it to be $= \frac{3Vm^2}{2(1-m)}\cos.^2\omega$, beginning from the quadrature where $\cos.\omega$ is 0. The mean value of this is when $\omega = \frac{\pi}{4}$, in which case it is $\frac{3Vm^2}{2(1-m)}\,\frac{1}{2}$. Therefore, the excess above the mean value is $\frac{3Vm^2}{4(1-m)}(2\cos.^2\omega - 1)$, that is, $\frac{3Vm^2}{4(1-m)}\cos.2\omega$. And we have for the whole value of the velocity

$$V\left\{1 + \frac{3m^2}{4(1-m)}\cos.2\omega\right\};$$

therefore, the mean value of the area is to its value at any other time,

$$\text{as } 1 \text{ to } 1 + \frac{3m^2}{4(1-m)} \cos.2\omega;$$

$$\text{or as } \frac{4(1-m)}{3m^2} \text{ to } \frac{4(1-m)}{3m^2} + \cos.2\omega.$$

But $m^2$ being $\frac{1}{178,75}$, it appears that $\frac{4(1-m)}{3m^2} = 220,46$; hence the area in a given small instant, is as $220,46 + \cos.2\omega$.

Thus the area at quadrature is to the area at syzygy,

as 219,46 to 221,46.

(Newton, Book III. Prop. xxviii.)

80. Prop. *To find the diameters of the oval orbit, into which the disturbing forces would convert the orbit of the moon, if circular when undisturbed.*

See Art. 64. Cor. 3. The oval orbit there spoken of will be perpendicular to the radius at syzygy and at quadrature; and by Art. 25, the force at such points is as $\frac{V^2}{R}$, where $R$ is the radius of curvature. Let $V$, $R$ correspond to the syzygy, $V'$, $R'$ to the quadrature. Then

$$\text{force at syzygy : force at quadrature} :: \frac{V^2}{R} : \frac{V'^2}{R'}.$$

Let 1 be the mean radius of the oval orbit, $1 - x$ the radius at syzygy, $1 + x$ at quadrature. Now, by last Proposition, the squares of the areas at syzygy and quadrature are in the ratio (neglecting the square of the smaller terms,)

$$1 + \frac{3m^2}{2(1-m)} : \ 1 - \frac{3m^2}{2(1-m)}.$$

and the velocities are as the area divided by the radius, and therefore their squares are as these quantities divided by

$(1-x)^2$ and $(1+x)^2$ respectively; or, neglecting again the squares and products of small quantities, $x$, &c,

$$V^2 : V'^2 :: 1 + 2x + \frac{3m^2}{2(1-m)} : 1 - 2x - \frac{3m^2}{2(1-m)}.$$

But the forces by which the moon is retained in her orbit at syzygy and quadrature, are the attraction of $T$ and $P$, together with the radial disturbing forces. Now, the attraction of $T$ and $P$ varies inversely as the square of the distance, and if 1 be its value at the mean distance 1,

$$\frac{1}{(1-x)^2} \text{ and } \frac{1}{(1+x)^2}, \text{ or } 1+2x \text{ and } 1-2x \text{ nearly,}$$

will be its values at syzygy and quadrature. The radial disturbing forces is at these points respectively

$$-2(F+G)m^2 \text{ and } (F+G)m^2;$$

and we may suppose $F+G$ to be equal to the mean force 1. Hence, the forces at syzygy and quadrature are

$$\text{as } 1 + 2x - 2m^2 : 1 - 2x + m^2.$$

Hence, we have, by comparing the ratio of the forces found in these different ways,

$$\frac{1 + 2x + \dfrac{3m^2}{2(1-m)}}{1 - 2x - \dfrac{3m^2}{2(1-m)}} \cdot \frac{R'}{R} = \frac{1 + 2x - 2m^2}{1 - 2x + m^2}.$$

Whence we find, neglecting all squares and products of $x$ and $m^2$,

$$\frac{R'}{R} = 1 - 3m^2\left(1 + \frac{1}{1-m}\right).$$

The ratio of $R'$ to $R$ will depend on the nature of the curve. Let it be supposed that we have an oval, of which $1+x$ and $1-x$ are the semiaxes, major and minor, $x$ being small: and $\theta$ being the angle which any radius vector drawn from the centre makes with the major axis, let the radius vector be $1 + x\cos.2\theta$. (This includes the ellipse and many

other ovals ultimately.) The oval which results from the disturbing forces cannot be such a figure as this, because the radius vector passes from its greatest to its least value, not in the course of an angle $\theta = \frac{\pi}{2}$, but in the course of an angle $\theta - \theta' = \frac{\pi}{2}$, or $\theta(1 - m) = \frac{\pi}{2}$. Let, therefore, the radius vector of the oval described by the moon be

$$r = 1 + x \cos. 2(1 - m)\theta,$$

which agrees with the condition just mentioned.

Then, at the extremity of the major axis, if we draw the subtense of the angle of contact, we have (*Introd.* Lemma XI.)

$$R' = \frac{\text{arc}^2}{\text{subtense}}.$$

And subtense = deflexion from tan. = semiaxis $- r \cos. \theta$

$$= 1 + x - \{1 + x \cos. 2(1 - m)\theta\} . \cos. \theta$$

$$= 1 - \cos. \theta + x \{1 - \cos. 2(1 - m)\theta . \cos. \theta\}.$$

Also, when $\theta$ is very small, $\cos. \theta = 1 - \frac{\theta^2}{2}$; hence, neglecting higher powers,

$$\text{subtense} = \frac{\theta^2}{2} + x\left(\frac{\theta^2}{2} + \frac{4(1 - m)^2\theta^2}{2}\right).$$

Also, arc $= (1 + x)\theta$; hence,

$$R' = \frac{2(1 + x)^2}{1 + x\{1 + 4(1 - m)^2\}}.$$

In like manner for $R$; let $\theta = \frac{\pi}{2(1 - m)} - \zeta$;

then $r = 1 - x \cos. 2(1 - m)\zeta$; and, as before,

$$\text{subtense} = 1 - x - \{1 - x \cos. 2(1 - m)\zeta\} \cos. \zeta$$

$$= 1 - \cos. \zeta - x\{1 - \cos. 2(1 - m)\zeta \cos. \zeta\}$$

$$= \frac{\zeta^2}{2} - x\left(\frac{\zeta^2}{2} + \frac{4(1-m)^2\zeta^2}{2}\right):$$

And the arc $= (1-x)\zeta$;

$$\text{hence, } R = \frac{\text{arc}^2}{\text{subtense}} = \frac{2(1-x)^2}{1 - x\{1 + 4(1-m)^2\}}.$$

Hence, we find that $R'$ is to $R$,

$$\text{as } \frac{(1+x)^2}{1 + x\{1 + 4(1-m)^2\}} \text{ to } \frac{(1-x)^2}{1 - x\{1 + 4(1-m)^2\}};$$

$$\text{therefore, } \frac{R'}{R} = \frac{(1+x)^2}{(1-x)^2} \cdot \frac{1 - x - 4x(1-m)^2}{1 + x + 4x(1-m)^2}$$

$$= \frac{1 + x - 4x(1-m)^2}{1 - x + 4x(1-m)^2} = 1 - 2x\{4(1-m)^2 - 1\},$$

neglecting higher powers.

Equating the two values of $\frac{R'}{R}$, we have

$$3m^2\left(1 + \frac{1}{1-m}\right) = 2x\{4(1-m)^2 - 1\},$$

$$x = \frac{\frac{3m^2}{2}\left(1 + \frac{1}{1-m}\right)}{4(1-m)^2 - 1}.$$

By putting for $m$ its value, we find $x = \frac{1}{139}$ nearly; and hence the distances of the moon at syzygy and quadrature, arising from the disturbing forces, are as 138 : 140, or as 69 : 70 nearly.

If we neglect $m^3$, $x$ is as $m^2$.

(Newton, Book III. Prop. xxix.)

81. Prop. *To find the Moon's Variation.*

The Variation (Art. 64,) is the error of longitude arising both from the acceleration of areas, and from the oval form of the orbit.

Fig. 158. Let $S$, $T$ be at rest, and $P$ revolve round $T$ in an *ellipse* $Cq$, describing areas proportional to the times. And let $CTp$ the area described in the same time, if the body revolve in a circle $Cp$ in the same period. Then, if

$$TC : TA :: 1 + x : 1 - x,$$

the angle $qTp$ will be the error of the longitude of $P$ arising from the oval form of the orbit, supposing the sun to be apparently at rest. And if we increase at the angles $CTq$, $CTp$, $CTA$ in the ratio $1 - m : 1$, so that $CTA = \frac{\pi}{2(1-m)}$, we shall have the motion of $p$ in an orbit, such as was supposed in the last Proposition; and the angle $qTp$ will become the error of longitude on that supposition. Also,

$$\tan. qTC = \frac{TA}{TC} \tan. pTC = \frac{1-x}{1+x} \tan. pTC.$$

In order to take account of the acceleration of areas, take $CTP$ such an angle, that

$$\tan. CTP = \sqrt{\frac{1-y}{1+y}} . \tan. CTq;$$

$1 - y : 1 + y$ being the ratio of the instantaneous area at syzygies to that at quadratures. This construction will give to $P$ the true motion. For, let $ATP = \omega$, $ATq = \phi$. Then,

$$\text{cotan. } \omega = \sqrt{\frac{1-y}{1+y}} \text{ cotan. } \phi;$$

and taking the instantaneous change,

$$\frac{\delta\omega}{\sin.^2 \omega} = \sqrt{\frac{1-y}{1+y}} . \frac{\delta\phi}{\sin.^2 \phi}; \quad \delta\omega = \sqrt{\frac{1-y}{1+y}} . \frac{\sin.^2 \omega}{\sin.^2 \phi} . \delta\phi.$$

Also, $$\frac{\cos.^2 \omega}{\sin.^2 \omega} = \frac{1-y}{1+y} . \frac{\cos.^2 \phi}{\sin.^2 \phi} = \frac{1-y}{1+y} \left\{ \frac{1}{\sin.^2 \phi} - 1 \right\};$$

whence $\dfrac{\sin.^2\omega}{\sin.^2\phi} = \sin.^2\omega + \dfrac{1+y}{1-y}\cos.^2\omega = \dfrac{1+y(\cos.^2\omega - \sin.^2\omega)}{1-y}$

$$= \frac{1 + y\cos.2\omega}{1-y}.$$

Hence, $\delta\omega = \delta\phi\,\dfrac{1 + y\cos.2\omega}{\sqrt{1-y^2}}$.

And by Art. 79, $y$ is $\dfrac{3m^2}{4(1-m)}$; also, $\delta\phi$ is constant in a given instant; therefore $\delta\omega$ is in the proportion of the instantaneous increment of the area proved in Art. 78: and $P$ is in its true position.

To find the error in longitude arising from the two causes above mentioned, let $ATp = \psi$; then

$$\text{cotan.}\,\phi = \frac{1-x}{1+x}\,\text{cotan.}\,\psi,$$

by the former part of the construction,

$$\text{cotan.}\,\omega = \frac{1-x}{1+x}\sqrt{\frac{1-y}{1+x}}\,\text{cotan.}\,\psi$$

$$= (1 - 2x - y)\,\text{cotan.}\,\psi,$$

neglecting powers of $x$ and $y$:

$$\text{hence, } \text{cotan.}\,\omega - \text{cotan.}\,\psi = -(2x+y)\,\text{cotan.}\,\psi,$$

or putting $\dfrac{\cos.}{\sin.}$ for cotan., $-\dfrac{\sin.(\omega-\psi)}{\sin.\omega\,\sin.\psi} = -(2x+y)\dfrac{\cos.\psi}{\sin.\psi}$;

$$\sin.(\omega-\psi) = (2x+y)\sin.\omega\cos.\psi = (2x+y)\sin.\omega\cos.\omega,$$

neglecting the difference of $\omega$ and $\psi$ after the first term.

Hence, $\omega - \psi = \left(x + \dfrac{y}{2}\right)\sin.2\omega$ or $\Delta\omega = \left(x + \dfrac{y}{2}\right)\sin.2\omega$.

If now, $\theta$ be the mean longitude of the moon, $m\theta + \beta$ that of the sun, as before; $\omega = m\theta - \theta + \beta$. And $\Delta\omega$ must depend on the angular distance of the sun and moon, which is proportional to the moon's mean longitude. Therefore,

$$\Delta\omega = -(1-m)\Delta\theta; \text{ and}$$

$$\Delta\theta = \frac{x + \frac{y}{2}}{1-m}\sin.2(\theta - m\theta - \beta);$$

and putting for $x$ and $y$ their values, we have the Variation of the Moon.

It appears by the reasoning, that the variation is as

$$\sin.2(\text{moon's mean long.} - \text{sun's mean long.}),$$

as already stated, Art. 64.

Taking only the first term of $x$, it is $m^2$, and $y = \frac{3m^2}{4}$.

Hence, $x + \frac{y}{2} = m^2 + \frac{3m^2}{8} = \frac{11m^2}{8}$, agreeing with the coefficient, as found analytically. (*Airy*, L. T. Art. 64).

(Newton, Book III. Prop. xxx.)

82. Prop. *To find the horary motion of the nodes in a circular orbit.*

Fig. 159. Let $S$ be the sun, $T$ the earth, $P$ the moon, $NPn$ the moon's orbit, $Nn$ being the line of nodes. The whole disturbing force of $S$ on $P$ has been resolved into two parts, (Art. 63); of which one, in the direction $PT$, does not draw $P$ from the plane of its orbit, and consequently does not effect the line of nodes; the other part, which call $Q$, urges the body $P$ in a direction parallel to $TS$, and consequently draws it from the plane of the orbit in which it is moving at $P$.

Let $PM$ represent the velocity of the moon independent of the disturbing force; and let $ML$, parallel to $TS$, represent

the velocity produced by the disturbing force in that direction in a unit of time (one hour).

The plane of $P$'s orbit is, by the action of the force $Q$, twisted from the position $PTM$ to the position $PTL$*; and if the plane $PTL$ meet the ecliptic in $Tl$, $mTl$ is the motion of the line of nodes in a unit of time (one hour).

The lines $ML$, $ml$ are in the same plane $LMPml$; but they can never meet, because $ML$ is parallel to the plane of the ecliptic; hence, $ML$, $ml$ are parallel.

Hence, we have

$$\frac{\sin.mTl}{\sin.mlT} = \frac{ml}{mT}, \quad \text{and } mTl = \frac{ml}{mT}\sin.mlT, \text{ nearly,}$$

$$(mTl \text{ being small}) = \frac{ml}{mT}\sin.STN. \quad \text{Now,}$$

$$\frac{ml}{mP} = \frac{ML}{MP}.$$

And $\frac{mP}{mT} = \sin.mTP$, (because $mPT$ is a right angle) $= \sin.PTN$.

$$\text{Therefore } \frac{ml}{mT} = \frac{ML}{MP}\sin.PTN.$$

$$\text{And } mTl = \frac{ML}{MP}\sin.PTN \,.\, \sin.STN.$$

Now, if $F$ be the central force by which $P$ is retained in its orbit, $3m^2F\cos.PTS$ (Art. 77,) is the force parallel to $TS$. Hence, if $V$ be the velocity, and $\delta t$ the time of describing $PM$, $PM = V\delta t$. Also, in the time $\delta t$, $LM$ = twice the space described = force $\times\, \delta t^2 = 3m^2F\delta t^2\cos.PTS$; whence,

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* The horary motion of the plane of $P$'s orbit depends on the direction in which $P$ is moving at the interval of one hour from its being at $P$; and the direction of the motion is rightly determined by taking $ML$ to represent the velocity generated by the force in one hour. $ML$ is double of the space described by the force in the same time. See *Introd.* Prop. 1. Note.

$$mTl = \frac{ML}{MP} = \frac{3m^2 F\delta t}{V}\cos.PTS \sin.PTN \sin.STN.$$

If $\theta$ be the longitude of the moon, $\theta'$ of the sun, $N$ of the nodes, $mTl$ is $\delta N$. Also, if 1 be the radius of $P$'s orbit, $F = V^2$; hence,

$$\delta N = 3m^2 V\delta t \cos.(\theta' - \theta)\sin.(\theta - N)\sin.(\theta' - N):$$

or since $\theta' = m\theta$, supposing $\theta'$ and $\theta$ to begin together;

and since $V\delta t = \delta\theta$,

$$\delta N = 3m^2 \delta\theta \cos.(\theta - m\theta)\sin.(\theta - N)\sin.(m\theta - N).$$

Cor. 1. If we suppose $\theta'$ and $N$ to remain constant for a whole revolution of $\theta$, we find the change of $N$ in one revolution of the moon, by integrating

$$3m^2 \cos.(\theta - \theta')\sin.(\theta - N)\sin.(\theta' - N)$$

for a whole circumference. Now,

$$\begin{aligned}\sin.(\theta - N) &= \sin.(\theta - \theta' + \theta' - N)\\ &= \sin.(\theta - \theta')\cos.(\theta' - N) + \cos.(\theta - \theta')\sin.(\theta' - N):\end{aligned}$$

and hence the expression to be integrated is

$$3m^2 \sin.^2(\theta' - N)\left\{\cos.(\theta - \theta')\sin.(\theta - \theta')\,\text{cotan}.(\theta' - N) + \cos.^2(\theta - \theta')\right\}.$$

The integral of $\sin.(\theta - \theta')\cos.(\theta - \theta')$ is $\frac{1}{2}\sin.^2(\theta - \theta')$; which, taken from $\theta - \theta' = 0$ to $\theta - \theta' = \pi$, is 0.

To find the integral of $\cos.^2(\theta - \theta')$, we have

$$\frac{d.\sin.x\cos.x}{dx} = \sin.^2 x - \cos.^2 x = 1 - 2\cos.^2 x,$$

$$\cos.^2 x = \frac{1}{2} - \frac{1}{2}\frac{d.\sin.x\cos.x}{dx};$$

$$\text{whence, } \int_x \cos.^2 x = \frac{x}{2} = \frac{\sin.x\cos.x}{2};$$

$$\text{and from } x = 0 \text{ to } x = \pi, \quad \int_x \cos.^2 x = \frac{\pi}{2}.$$

Hence, for a semi-circumference of $\theta - \theta'$, we have the motion of $N$

$$= \frac{3m^2\pi}{2} \sin.^2(\theta' - N).$$

Cor. 2. When the moon is at syzygy, so that $\theta = \theta'$, we have for the instantaneous motion of the node

$$\delta N = 3m^2 \delta\theta \sin.^2(\theta' - N);$$

and if this were to continue at the same rate for a whole semicircle, we should have to put $\pi$ instead of $\delta\theta$, and the motion would be

$$3m^2\pi \sin.^2(\theta' - N).$$

Hence, in any given position of the nodes, the mean instantaneous (or horary) motion of the node is half the instantaneous (or horary) motion which takes place when the moon is in syzygy.

Cor. 3. When the nodes are in quadrature, we have $\theta' - N = \frac{\pi}{2}$; and the motion in a whole revolution of the moon will be $3m^2\pi$.

Cor. 4. If we take the equation

$$\delta N = -8m^2\delta\theta \cos.(\theta - m\theta)\sin.(\theta - N)\sin.(m\theta - N),$$

and resolve the products of sines and cosines into the sums and differences of such quantities, (which is always possible,) we find

$$\delta N = -\frac{3m^2}{4}\delta\theta\{1 + \cos.(2\theta - 2m\theta) - \cos.(2\theta - 2N) - \cos.(2m\theta - 2N)\},$$

which may be integrated as a differential equation between $N$ and $\theta$, and gives, making $\frac{3m^2}{4} = i$,

$$N = -\frac{3m^2}{4}\theta - \frac{3m^2}{8(1-m)}\sin.(2\theta - 2m\theta) + \frac{3m^2}{8(1-i)}\sin.(2\theta - 2N)$$

$$+ \frac{3m^2}{8(m-i)}\sin.(2m\theta - 2N).$$

Here the last term is considerable, on account of the smallness of its divisor, $8m-8i$, or $8m-6m^2$; it gives the annual equation of the motion of the node. (*Laplace*, Mec. Cel. T. v. p. 376.)

(NEWTON, Book III. Prop. XXXI.)

83. PROP. *To find the horary motion of the nodes of the moon in the elliptical orbit assumed in Art.* 79.

Fig. 160. Let $QPMAq$ be the circular orbit, $Qpmaq$ the elliptical orbit; $p$ the moon, $S$ the sun, $T$ the earth as before. Let $pm$ be an arc described in the instant $\delta t$, $N$ and $n$ the nodes, $PpK$, $Mmk$ perpendiculars in the line of quadrature $Qq$, (which is the major axis of the ellipse,) and let these meet $Nn$ in $D$, $d$. Let the moon $p$ describe areas proportional to the times; then the motion of the nodes in one revolution will be diminished in the ratio $Dp : DP$, by the elliptical form of the orbit.

To prove this, let $PF$ be a tangent to the circle in $P$, meeting $TNF$; $pf$ a tangent to the ellipse in $p$, meeting $TN$ in $f$; these tangents will meet $TQ$ in the same point $Y$. Let $LM$ be the effect of the disturbing force at $P$ in the circular orbit, as in last Prop., and $lm$ the effect of the disturbing force at $p$; produce $LP$, $lp$ to meet the plane of the ecliptic in $G$, $g$; join $FG$, $fg$, and let these produced meet $pf$, $pg$ and $TQ$ in $c$, $e$ and $R$ respectively; and let $fg$ produced meet $TQ$ in $r$. Then $fTg$, $FTG$ will be the motion of the nodes in the circular and elliptical orbits respectively.

The force in the circular orbit is to the force in the elliptical orbit as $PK$ to $pK$. For (Cor. 7, Art. 66,) if $\frac{S}{ST^2}$ be the force of $S$ on $T$, $\frac{3S.PK}{ST^3}$ is the force parallel to $TS$.

$$\text{Hence, } \frac{ml}{ML} = \frac{pK}{PK} = \frac{cR}{FR}.$$

$$\text{But } \frac{FG}{ML} = \frac{FP}{LP} = \frac{cp}{lp} = \frac{ce}{lm};$$

$$\text{whence } \frac{tm}{ML} = \frac{ce}{FG}, \text{ and } \frac{ce}{FG} = \frac{cR}{FR}.$$

$$\text{Now, } \frac{fg}{FG} = \frac{fg}{ce} \cdot \frac{ce}{FG} = \frac{fg}{ce} \cdot \frac{cR}{FR} = \frac{fp}{cp} \cdot \frac{cR}{FR}.$$

$$\text{Also, } \frac{FT}{fT} = \frac{FR}{fr} = \frac{FR}{cR} \cdot \frac{cR}{fr} = \frac{FR}{cR} \cdot \frac{cY}{fY},$$

$$\text{therefore, } \frac{fg}{fT} \cdot \frac{FT}{FG} = \frac{fg}{FG} \cdot \frac{FT}{fT} = \frac{fp}{cp} \cdot \frac{cY}{fY} = \frac{fp}{fY} \cdot \frac{cY}{cp};$$

or, drawing $ph$ parallel to $TN$, meeting $PY$,

$$\frac{\text{angle } fTg}{\text{angle } FTG} = \frac{fp}{fY} \cdot \frac{cY}{cp} = \frac{Fh}{FY} \cdot \frac{FY}{FP} = \frac{Fh}{FP} = \frac{Dp}{DP}.$$

Cor. We have $DK = TK \tan. DTK$; or, $TP$ being 1,

$$DK = \sin. ATP \,.\, \text{cotan.}\, ATn = \sin. (\theta - \theta') \text{ cotan.} (\theta - N).$$

$$\text{Also, } DP = TP \,.\, \frac{\sin. DTP}{\sin. PDT} = \frac{\sin. (\theta - N)}{\sin. (\theta' - N)};$$

$$\text{and } \frac{Kp}{KP} = \frac{aT}{AT} = \frac{1 - x}{1 + x}, \text{ as in Art. 79.}$$

$$\text{Hence, } \frac{Dp}{DP} = \frac{DK + Kp}{DP} = \frac{1}{DP} \left\{DK + \frac{1 - x}{1 + x} KP\right\}$$

$$= \frac{\sin. (\theta' - N)}{\sin. (\theta - N)} \left\{\sin. (\theta - \theta') \text{ cotan.} (\theta - N) + \frac{1 - x}{1 + x} \cos. (\theta - \theta')\right\};$$

whence $\delta N$ is found from the expression in last Proposition, (p. 144,) by multiplying by this factor.

We find thus

$$\delta N = 3m^2 \delta\theta \sin.^2 (\theta' - N)$$

$$\left\{\cos. (\theta - \theta') \sin. (\theta - \theta') \text{ cotan.} (\theta - N) + \frac{1 - x}{1 + x} \cos.^2 (\theta - \theta')\right\}.$$

Integrating for a semi-circumference of $\theta - \theta'$, as in Cor. 1, Art. 82, we shall have the same result as before, with the exception of the factor $\dfrac{1-x}{1+x}$ in the latter term.

Therefore, for a revolution of $p$, we have the motion of $N$

$$= \frac{1-x}{1+x} \cdot \frac{3m^2\pi}{2} \sin.^2(\theta' - N).$$

A correction is also due to the acceleration of areas. (*Newton*, Vol. III. p. 428.)

(Newton, Book III. Prop. XXXII.)

84. Prop. *To find the Mean Motion of the Moon's Nodes.*

If $\delta N$ now represent the motion of the node in any instant, at the rate of the *mean motion for one revolution of the moon,* (*the monthly mean,*) we have

$$\delta N = -\frac{3m^2}{2}\delta\theta \sin.^2(\theta' - N) \text{ Art. 81, Cor. 1,}$$

$$= -\frac{3m^2}{2}\delta\theta \sin.^2(m\theta - N).$$

Let $m\theta - N = \phi$, therefore $m\delta\theta = \delta\phi + \delta N$,

$$\delta N = -\frac{3m}{2}\{\delta\phi + \delta N\} \sin.^2\phi$$

$$\delta N = -\frac{\frac{3m}{2}\sin.^2\phi \cdot \delta\phi}{1 + \frac{3}{2}m\sin.^2\phi}.$$

We have to integrate

$$-\frac{\frac{3m}{2}\sin.^2\phi}{1 + \frac{3m}{2}\sin.^2\phi} = -\frac{3m}{2}\sin.^2\phi + \frac{\frac{3m}{2}\sin.^4\phi}{\frac{2}{3m} + \sin.^2\phi}.$$

The integral of $-\frac{3m}{2}\sin.^2\phi$, for a whole revolution, is $-\frac{3m}{4}.2\pi$, as in Art. 81, Cor. 1. The other term is

$$\frac{9m^2}{4}\frac{\sin.^4\phi}{1+\frac{3m}{2}\sin.^2\phi} = \frac{9m^2}{4}\left\{\sin.^4\phi - \frac{3m}{2}\sin.^6\phi\right\} \text{ nearly.}$$

Now, for a whole circumference,

$$\int_\phi \sin.^4\phi = \frac{3}{8}.2\pi, \quad \int_\phi \sin.^6\phi = \frac{5}{16}.2\pi.$$

Hence, the whole term is

$$\frac{9m^2}{4}\left\{\frac{3}{8} - \frac{15}{32}m\right\}2\pi = \frac{27}{32}m^2\left\{1 - \frac{5m}{4}\right\}2\pi.$$

And the whole motion of the node, *in one revolution of the node,*

$$-\frac{3m}{4}2\pi\left\{1 - \frac{9m}{8} + \frac{45}{32}m^2\right\}.$$

This is the motion of the node in one revolution of the node from the sun to the sun again. Let this $= -2F\pi$: therefore this is the motion of the node while the sun describes $2\pi(1-F)$: and therefore in a year, while the sun describes $2\pi$, the mean motion will be greater in the ratio $\frac{1}{1-F}$. This gives for the annual motion of the node,

$$-\frac{3m}{4}2\pi\left\{1 - \frac{3m}{8} + \frac{9m^3}{32}\right\}.$$

The mean motion in one revolution of the moon is this $\times\, m$, that is

$$-2\pi\left(\frac{3m^2}{4} - \frac{9m^3}{32} + \frac{27m^4}{64}\right);$$

agreeing, as far as the term involving $m^3$, with the expression obtained analytically. (*Airy*, L. T. Art. 72.)

(Newton, Book III. Prop. xxxiv.)

85. Prop. *To find the horary variation of the inclination of the moon's orbit to the plane of the ecliptic.*

Fig. 161. Let $A$ and $a$ be the syzygies, $Q$, $q$, the quadratures; $N$, $n$, the nodes; $P$ the place of the moon; $Pp$ a perpendicular on the plane of the ecliptic; $mTl$ the motion of the nodes in the instant $\delta t$, as determined above.

Let $PG$, $pG$ be perpendicular to $Tm$, and join $Pg$. $PGp$ is the inclination of the orbit when the moon is at $P$, and at the end of the next instant the inclination is $Pgp$, and the difference $GPg$ is the variation of the inclination. The deviation of $P$, during this instant, from the plane $PgT$, will not sensibly affect the inclination. For if $ML$ be the deviation from this plane, as in page 143, we have

$$Gg = TG \, . \, mTl = TG \frac{ML}{MP} \sin. PTN \sin. STN.$$

Hence as $PM$ in fig. 159 becomes smaller and smaller, $Gg$ becomes larger and larger compared with $ML$. Hence the angle which $ML$ subtends from $g$, when the instant is small, is indefinitely less than the angle which $Gg$ subtends from $P$: therefore the latter angle $GPg$ may be taken to represent the whole change of the inclination.

$$\text{Now } GPg = \frac{Gg \, . \sin. PGp}{PG} = \frac{TG \, . \sin. PGp}{PG} \text{ang.} mTl:$$

and, putting for $mTl$ its value, (p. 143,)

$$\frac{ML}{MP} \sin. PTN \sin. STN, \text{ and for } \frac{TG}{PG}, \ \frac{\cos. PTN}{\sin. PTN};$$

$$GPg = \frac{ML}{MP} \cos. PTN \sin. STN \sin. PGp.$$

Also, as in Art. 81.

$ML = 3m^2 F \, . \, \delta t^2 . \cos. PTS$; and if $TP = 1$, $F = V^2$, $MP = V\delta t$. Hence

$$GPg = 3m^2 . \, V\delta t \, . \cos. PTS \, . \cos. PTN \sin. STN \sin. PGp.$$

If $\gamma$ be the inclination of the orbit, $\theta$, $\theta'$, $N$, as before, we have $V\delta t = \delta\theta$, and

$$\delta\gamma = -3m^2 . \delta\theta \sin.\gamma . \cos.(\theta - N) \sin.(\theta' - N) \cos.(\theta - \theta').$$

In consequence of the ellipticity of the orbit, the mean motion in one revolution of $P$ is diminished in the ratio $1 - x : 1 + x$: (Art. 83:) and the variation of the inclination will be diminished in the same proportion.

Cor. 1. Supposing $\theta'$, and also $N$ and $\gamma$, to be constant for a revolution of $P$, we may integrate in the same manner as in Art. 82, Cor. 1; and we find thus for the change of inclination during a semi-circumference described by the moon,

$$\frac{3m^2\pi}{2} \sin.\gamma . \sin.(\theta' - N) \cos.(\theta' - N);$$

$$\text{or,} \quad \frac{3m^2\pi}{4} \gamma . \sin.(2\theta' - 2N), \quad \text{since } \gamma \text{ is small.}$$

Cor. 2. If $\theta' - N = \phi$, the distance of the sun from the node; then if $\delta\phi$ be any instantaneous increment of this angle, and if we suppose $\delta\gamma$ to be the corresponding change of the inclination with the *monthly mean* rate of change, by Cor. 1.

$$\delta\gamma = \frac{3m^2}{4} \delta\theta . \gamma . \sin.(2\theta' - 2N).$$

$$\text{Let } \phi = \theta' - N = m\theta - N, \text{ hence } \delta\theta = \frac{\delta\phi}{m},$$

$$\text{and } \delta\gamma = \frac{3m}{4} \delta\phi . \gamma . \sin.2\phi.$$

Cor. 3. When the nodes are in quadratures,

$$\theta' - N = \frac{\pi}{2}, \quad \cos.(\theta - \theta') = \sin.(\theta - N);$$

$$\delta\gamma = -3m^2\delta\theta \sin.\gamma \cos.(\theta - N) \sin.(\theta - N);$$

and integrating from quadrature, that is from $\theta - N = 0$, the change of inclination is

$$-\frac{3m^2}{2}\sin.\gamma\sin.^2(\theta-N);$$

and at syzygy the change is

$$-\frac{3m^2}{2}\sin.\gamma.$$

(Newton, Book III. Prop. xxxv.)

86. Prop. *To find the Inclination of the Moon's Orbit at a given time.*

Fig. 162. Make the following construction. Let $AC = 1$, represent the mean value of the inclination; $CB$, $CD$ the greatest variation on each side of this mean.

Let $CB = CD = \beta$, and take $CE = \epsilon$, such that $2\epsilon - \epsilon^2 = \beta^2$: with centre $C$ and radius $CB$, describe a circle, and at the point $E$, make the angle $BEG = 2\phi$, in Cor. 2. last Prop. Draw $GH$ perpendicular to $AC$; then $AH$ represents the inclination.

Let $EG = \rho$; then by the triangle $GCE$,

$$\beta^2 = \rho^2 + \epsilon^2 + 2\rho\epsilon\cos.2\phi;$$

or, putting for $\beta^2$ its value,

$$2\epsilon - \epsilon^2 = \rho^2 + \epsilon^2 + 2\rho\epsilon\cos.2\phi;$$

whence we find

$$\frac{\rho^2}{2\epsilon} = 1 - \epsilon - \rho\cos.2\phi = AC - CE - EH = AH.$$

$$\text{Hence,}\ \delta.AH = \frac{\rho}{\epsilon}\delta\rho.$$

$$\text{But since}\ \beta^2 = \rho^2 + \epsilon^2 + 2\rho\epsilon\cos.2\phi,$$

we have

$$\delta\rho = \frac{2\rho\epsilon\sin.2\phi.\delta\phi}{\rho+\epsilon\cos.2\phi}\ \text{whence}\ \delta.AH = \frac{2\rho^2\sin.2\phi.\delta\phi}{\rho+\epsilon\cos.2\phi}.$$

Also by the equation

$$2\epsilon - \epsilon^2 = \rho^2 + \epsilon^2 + 2\rho\epsilon \cos.2\phi,$$

we find

$$\rho + \epsilon \cos.2\phi = \sqrt{(2\epsilon - 2\epsilon^2 + \epsilon^2 \cos.^2 2\phi)}$$

$$= \sqrt{2\epsilon}.\{1 - \frac{\epsilon}{2} + \frac{\epsilon}{4}\cos.^2 2\phi + \&c.\};$$

hence this is very nearly constant, for $\epsilon$ is very small.

Therefore, $\delta.AH = \frac{2\rho^2 \sin.2\phi.\delta\phi}{\sqrt{2\epsilon}} = \frac{\rho^2}{2\epsilon}.2\sqrt{2\epsilon}.\sin.2\phi.\delta\phi;$

and $\frac{\rho^2}{2\epsilon}$ is $AH$. Hence if $AH$ be $s$, we have

$$\delta s = 2\sqrt{2\epsilon}.\delta\phi.s.\sin.2\phi;$$

agreeing in its form with the equation in Cor. 2. Art. 85. Also at the greatest and least values, $AH$ is proportional in quantity to the greatest and least inclination: therefore $AH$ is always proportional to the inclination.

Since $2\epsilon - \epsilon^2 = \beta^2$, $\epsilon = 1 - \sqrt{1 - \beta^2} = \frac{\beta^2}{2}$ nearly, because $\beta$ is small; whence

$$\delta s = 2\beta.\delta\phi.s.\sin.2\phi.$$

(Here $\phi$ is supposed to increase uniformly by the motion of the sun and node, which is not exact.)

Let $BEG$ $(2\phi)$ be a right angle, or

$$2\theta' - 2N = \frac{\pi}{2}, \text{ then } \delta s = \beta.s.\delta\phi;$$

and the change of inclination in this case, in one revolution of the moon, is known by Cor. 1. of Art. 85,

$$\text{and is } \frac{3m^2\pi}{4}s = \frac{3m}{8}s.2\pi m.$$

Now while $\theta$ increases by $2\pi$ as it does in Art. 85, Cor. 1, $\phi$ increases by $2\pi m$. Hence it appears that $\beta$ is $\frac{3m}{8}$; and $\delta s = \frac{3m}{4}\delta\phi . s . 2\phi$, as already found Art. 85, Cor. 2.

The whole change of inclination is between the limits $1-\beta$ and $1+\beta$, or $1-\frac{3m}{8}$ and $1+\frac{3m}{8}$, so far as the effect of the mean motion for a revolution of the moon is concerned. Hence if $k$ be the mean inclination, $k\left(1-\frac{3m}{8}\right)$ is the least inclination, so far as the monthly mean affects it.

But besides the effect of the monthly mean, the inclination is diminished or increased during the month. If the nodes be in quadratures, the diminution while the moon passes from quadrature to syzygy is $s . \frac{3m^2}{2}$, as shewn in Cor. 3. Art. 85.

$$\text{Hence } k\left(1-\frac{3m}{8}-\frac{3m^2}{8}\right), \quad k\left(1+\frac{3m}{8}+\frac{3m^2}{8}\right)$$

will be limits of the inclination, when we consider the moon in syzygies;

$$\text{and } k\left(1-\frac{3m}{8}+\frac{3m^2}{8}\right), \quad k\left(1+\frac{3m}{8}-\frac{3m^2}{8}\right)$$

will be the limits when we consider the moon in quadratures.

If the nodes pass from syzygies to quadratures, and the moon come into syzygies, the inclination passes from

$$k\left(1+\frac{3m}{8}+\frac{3m^2}{8}\right) \text{ to } k\left(1-\frac{3m}{8}+\frac{3m^2}{8}\right)$$

by the effect of the monthly means, and receives a diminution of $\frac{3m^2}{2}$ in its passage from quadrature to syzygy; of which

diminution, one half, $\frac{3m^2}{4}$, belongs to the monthly mean, and the remainder $\frac{3m^2}{4}$ diminishes the inclination to $k\left(1-\frac{3m}{8}-\frac{3m^2}{8}\right)$.

If $\gamma$ be the longitude of the node, $\beta$ of the sun, we have by the analytical method, (*Airy*, L. T. Art. 68.)

$$s = k\left\{1+\frac{3m}{8}\cos.2(\gamma-\beta)\right\};$$

whence it appears that the coefficient of the first term of the variation of the inclination agrees with our result.

## *Sect. IV.* Analytical Methods of the Solution of the Problem of three or more Bodies.

### THEORY OF THE PLANETS.

87. Prop. *To find the general equations of motion of any number of small bodies revolving about a large one, and attracting each other with forces which vary inversely as the square of the distance.*

Let $M$ be the mass of the central body, $m$, $m'$, $m''$, &c. the masses of the other bodies. The attraction of any one of the bodies, as $m$, upon any other, at any distance, is supposed to be $\frac{m}{(\text{distance})^2}$.

Let the bodies be referred to three rectangular co-ordinates, and let $x$, $y$, $z$, be the co-ordinates of $m$; $x'$, $y'$, $z'$, those of $m'$; and so on. Then the distance of $m$ and $m'$ is

$$\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}};$$

and the attraction of $m'$ on $m$ is

$$\frac{m'}{(x'-x)^2+(y'-y)^2+(z'-z)^2}.$$

If this force be resolved in the direction of the co-ordinates $x$, $y$, $z$, it will easily be seen that the resolved part in the direction of $x$, tending to increase $x$, will be,

$$\frac{m'(x'-x)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}},$$

which is $\dfrac{1}{m}\cdot\dfrac{d\,\dfrac{mm'}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}}}}{dx}$.

In like manner the action of $m''$ on $m$, resolved parallel to $x$ will be

$$\frac{1}{m}\,\frac{d\,\dfrac{mm''}{\{(x''-x)^2+(y''-y)^2+(z''-z)^2\}^{\frac{1}{2}}}}{dx}.$$

Assume $\lambda = \dfrac{mm'}{\{(x''-x')^2+(y''-y')^2+(z''-z')^2\}^{\frac{1}{2}}}$

$$+\frac{mm''}{\{(x''-x)^2+(y''-y)^2+(z''-z)^2\}^{\frac{1}{2}}}$$

$$+\frac{m'm''}{\{(x''-x')^2+(y''-y')^2+(z''-z')^2\}^{\frac{1}{2}}}+\text{\&c.}$$

$\lambda$ being thus the sum of the products of each combination of two, of the bodies $m$, $m'$, $m''$, &c., divided by their respective distance. We then have

$$\frac{1}{m}\left(\frac{d\lambda}{dx}\right),$$

the sum of the action of all the bodies $m'$, $m''$, &c. upon $m$, resolved parallel to $x$, and acting from the origin; $\left(\dfrac{d\lambda}{dx}\right)$ representing the partial differential coefficient of the function $\lambda$ with regard to $x$ alone.

In like manner

$$\frac{1}{m}\left(\frac{d\lambda}{dy}\right),\quad \frac{1}{m}\left(\frac{d\lambda}{dz}\right)$$

would represent the action on $m$ resolved parallel to $y$, and to $z$, respectively.

$$\text{And } \frac{1}{m'}\left(\frac{d\lambda}{dx'}\right), \quad \frac{1}{m'}\left(\frac{d\lambda}{dy'}\right), \quad \frac{1}{m'}\left(\frac{d\lambda}{dz'}\right),$$

would in like manner represent the action of all the bodies $m$, $m''$, &c. on $m'$; and so of the rest.

Let $X$, $Y$, $Z$ be the co-ordinates of $M$: and now let
$X+x$, $Y+y$, $Z+z$, be the co-ordinates of $m$,
$X+x'$, $Y+y'$, $Z+z'$,............... those of $m'$ &c.
then it is clear that the distance of $m$ and $m'$ will still be

$$\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}}$$

as before: and $\lambda$ being still assumed the same function of $m$, $x$, $x'$ &c., the resolved forces will still be exprest in terms of its partial differential coefficients. Also let $r$, $r'$, $r''$, &c., be the distances of $m$, $m'$, $m''$, &c. from $M$; so that

$$r=(x^2+y^2+z^2)^{\frac{1}{2}}, \quad r'=(x'^2+y'^2+z'^2)^{\frac{1}{2}}, \text{ \&c.}$$

The action of $m$ on $M$, resolved parallel to the axis of $x$, acting to the origin, will be $\frac{mx}{r^3}$; that of $m'$ on $M$, resolved in the same direction, is $\frac{m'x}{r'^3}$; and so of the rest: and similarly for the other two co-ordinates. Hence, if $\Sigma$ indicate the sum of all the terms resulting from $m$, $m'$, $m''$, &c., we have, collecting all the forces which act on $M$,

$$0=\frac{d^2X}{dt^2}-\Sigma\frac{mx}{r^3};$$

$$0=\frac{d^2Y}{dt^2}-\Sigma\frac{my}{r^3};$$

$$0=\frac{d^2Z}{dt^2}-\Sigma\frac{mz}{r^3}.$$

The action of $M$ on $m$, resolved parallel to $x$, and acting from the origin, is $-\frac{Mx}{r^3}$; and the sum of the actions of all the

bodies $m'$, $m''$, &c., on $m$, is $\frac{1}{m}\left(\frac{d\lambda}{dx}\right)$; hence for the motion of $m$,

$$0 = \frac{d^2(X+x)}{dt^2} + \frac{Mx}{r^3} - \frac{1}{m}\left(\frac{d\lambda}{dx}\right),$$

or, since $\frac{d^2(X+x)}{dt^2} = \frac{d^2X}{dt^2} + \frac{d^2x}{dt^2} = \Sigma\frac{mx}{r^3} + \frac{d^2x}{dt^2}$, we find

$$0 = \frac{d^2x}{dt^2} + \frac{Mx}{r^3} + \Sigma.\frac{mx}{r^3} - \frac{1}{m}\left(\frac{d\lambda}{dx}\right); \text{ similarly,}$$

$$0 = \frac{d^2y}{dt^2} + \frac{My}{r^3} + \Sigma.\frac{my}{r^3} - \frac{1}{m}\left(\frac{d\lambda}{dy}\right);$$

$$0 = \frac{d^2z}{dt^2} + \frac{Mz}{r^3} + \Sigma.\frac{mz}{r^3} - \frac{1}{m}\left(\frac{d\lambda}{dz}\right).$$

Now, make

$$R = \frac{m'(xx'+yy'+zz')}{r'^3} + \frac{m''(xx''+yy''+zz'')}{r''^3} + \text{\&c.} - \frac{\lambda}{m},$$

and we have

$$\left(\frac{dR}{dx}\right) = \frac{m'x}{r'^3} + \frac{m''x''}{r'^3} + \text{\&c.} - \frac{1}{m}\left(\frac{d\lambda}{dx}\right) = \Sigma\frac{mx}{r^3} - \frac{mx}{r^3} - \frac{1}{m}\left(\frac{d\lambda}{dx}\right),$$

$$\text{or } \frac{mx}{r^3} + \left(\frac{dR}{dx}\right) = \Sigma\frac{mx}{r^3} - \frac{1}{m}\left(\frac{d\lambda}{dx}\right).$$

Substituting for the latter two terms their value, in the first of the above three equations, and putting $\mu$ for $M+m$, in the result, we find

$$\left.\begin{aligned} 0 &= \frac{d^2x}{dt^2} + \frac{\mu x}{r^3} + \left(\frac{dR}{dx}\right); \\ \text{similarly } 0 &= \frac{d^2y}{dt^2} + \frac{\mu y}{r^3} + \left(\frac{dR}{dy}\right); \\ 0 &= \frac{d^2z}{dt^2} + \frac{\mu z}{r^3} + \left(\frac{dR}{dz}\right). \end{aligned}\right\} (P).$$

The investigations of the planetary theory consist in obtaining from these equations various integrals.

88. Prop. *To find two integrals of equations* (P).

Multiply the three equations respectively by

$$2\frac{dx}{dt},\ 2\frac{dy}{dt},\ 2\frac{dz}{dt},$$

add and integrate; and we have

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 - \frac{2\mu}{r} + \frac{\mu}{a} + 2\int_t \frac{d(R)}{dt} = 0; \ \ldots (Q).$$

For the second terms of each of the three equations give

$$\frac{\mu}{r^3}\left(x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right),$$

which is equal to $\frac{\mu}{r^2}\frac{dr}{dt}$, because $x^2 + y^2 + z^2 = r^2$. And the integral of this is $-\frac{2\mu}{r} + \frac{\mu}{a}$, $2a$ being the value of $r$ for which the integral vanishes.

Also, $R$ being a function of $x$, $y$, $z$, which are themselves functions of $t$, if we differentiate with regard to $x$, $y$, $z$ only,

$$\frac{d(R)}{dt} = \left(\frac{dR}{dx}\right)\frac{dx}{dt} + \left(\frac{dR}{dy}\right)\frac{dy}{dt} + \left(\frac{dR}{dz}\right)\frac{dz}{dt}.$$

Hence the above equation (Q) is true.

Again, we find $\frac{1}{2}\frac{d^2 . r^2}{dt^2} = \frac{1}{2}\frac{d\frac{d . r^2}{dt}}{dt}$

$$= \frac{d\left(x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right)}{dt}$$

$$= x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2} + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

Hence, multiply the three equations ($P$) by $x$, $y$, $z$ and add them to the integral $Q$, and we have

$$\frac{1}{2}\frac{d^2.r^2}{dt^2}-\frac{\mu}{r}+\frac{\mu}{a}+2\int_t\frac{d(R)}{dt}$$

$$+x\left(\frac{dR}{dx}\right)+y\left(\frac{dR}{dy}\right)+z\left(\frac{dR}{dz}\right)=0;\ \ldots(R)$$

observing that $\dfrac{x^2+y^2+z^2}{r^3}=\dfrac{1}{r}$.

$R$ is here to be differentiated with regard to the place of $m$ only.

If the motion take place entirely in the plane of $x$, $y$, so that $z$ disappears, we have $x=r\cos.\theta$, $y=r\sin.\theta$, $\theta$ being the angle which $r$ makes with $x$. Hence

$$\left(\frac{dR}{dx}\right)=\left(\frac{dR}{dr}\right)\frac{dr}{dx}+\left(\frac{dR}{d\theta}\right)\frac{d\theta}{dx},$$

$$\left(\frac{dR}{dy}\right)=\left(\frac{dR}{dr}\right)\frac{dr}{dy}+\left(\frac{dR}{d\theta}\right)\frac{d\theta}{dy};$$

$$r=\sqrt{(x^2+y^2)},\quad \theta=\tan.^{-1}\frac{y}{x}.$$

therefore $\dfrac{dr}{dx}=\dfrac{x}{r}$, $\dfrac{dr}{dy}=\dfrac{y}{r}$, $\dfrac{d\theta}{dx}=-\dfrac{y}{x^2+y^2}$, $\dfrac{d\theta}{dy}=\dfrac{x}{x^2+y^2}$.

Hence,

$$x\left(\frac{dR}{dx}\right)+y\left(\frac{dR}{dy}\right)\ \text{becomes}\ \left(\frac{dR}{dr}\right)\left(\frac{x^2+y^2}{r}\right)=r\left(\frac{dR}{dr}\right),$$

and the equation ($R$) becomes

$$\frac{1}{2}\frac{d^2.r^2}{dt^2}-\frac{\mu}{r}+\frac{\mu}{a}+2\int_t\frac{d(R)}{dt}+r\left(\frac{dR}{dr}\right)=0\ldots(R').$$

$R$ being differentiated with regard to the place of $m$ only.

This equation serves to determine the change produced in $r$, and from thence the disturbance of the place of the body, produced by the forces exprest in the function $R$. The calculation

of these perturbations however requires us to expand the function $R$, and to collect the results, by a long and peculiar set of processes; and for these we must refer the reader to other works. See *Airy's* Tracts, Planetary Theory, Art. 83. *Mrs Somerville*, Chap. IX. Mecanique Celeste. I. Partie, Liv. II.

## LUNAR THEORY.

89. PROP. *A body* P *revolves about another body* T *by the action of a central force, and is moreover acted on by the disturbing forces arising from the mutual attractions of the body* S; *to find the equations of the motion of* P.

Fig. 163. Let the body $P$ be referred to three rectangular co-ordinates, $x$, $y$, $z$, of which $T$ is the origin. Let $TE$ be the axis of $x$; $Pp$ a perpendicular on the plane of $xy$, and $pm$ parallel to the axis of $y$, so that $Tm = x$, $mp = y$, $pP = z$. Let $pq$ be perpendicular to $Tp$; let the forces which act upon $P$ be resolved in the directions of the three lines $pT$, $pq$, $pP$, and let them be, in these three directions $P$, $T$, $S$, respectively.

Then if $ETp = \theta$, it is easily seen that

a force $P\cos.\theta + T\sin.\theta$ tends to diminish $x$,

a force $P\sin.\theta - T\cos.\theta$ tends to diminish $y$,

and a force $S$ tends to diminish $z$.

Hence we have, if $TP = \rho$, and $s = \tan.PTp$;

$$x = \rho\cos.y, \quad y = \rho\sin.\theta, \quad z = \rho s.$$

$$\frac{d^2x}{dt^2} = -P\frac{x}{\rho} - T\frac{y}{\rho},$$

$$\frac{d^2y}{dt^2} = -P\frac{y}{\rho} + T\frac{x}{\rho},$$

$$\frac{d^2z}{dt^2} = -S.$$

Hence $2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}+2\frac{dy}{dt}\cdot\frac{d^2y}{dt^2}=-\frac{2P}{\rho}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)$

$$+\frac{2T}{\rho}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right).$$

But $2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}+2\frac{dy}{dt}\cdot\frac{d^2y}{dt^2}=\frac{d}{dt}\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}$

$$=\frac{d}{dt}\left\{\left(\frac{d\rho}{dt}\right)^2+\rho^2\left(\frac{d\theta}{dt}\right)^2\right\}.$$

And $x\frac{dx}{dt}+y\frac{dy}{dt}=\rho\frac{d\rho}{dt}$; $x\frac{dy}{dt}-y\frac{dx}{dt}=\rho^2\frac{d\theta}{dt}$: whence

$$\frac{d}{dt}\left\{\left(\frac{d\rho}{dt}\right)^2+\rho^2\left(\frac{d\theta}{dt}\right)^3\right\}=-2P\cdot\frac{d\rho}{dt}+2T\rho\cdot\frac{d\theta}{dt}\ldots\ldots(a).$$

Also we find $x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=T\frac{x^2+y^2}{\rho}=T\rho.$

But $x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=\frac{d}{dt}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=\frac{d}{dt}\left(\rho^2\cdot\frac{d\theta}{dt}\right)$;

therefore, $\frac{d}{dt}\left(\rho^2\cdot\frac{d\theta}{dt}\right)=T\rho\ldots\ldots\ldots\ldots(b).$

Also, we have $\frac{d^2(\rho s)}{dt^2}=-S\ldots\ldots\ldots\ldots(c).$

In these equations, $(a)$, $(b)$, $(c)$, $t$ is to be eliminated, and then by the expansion of the expressions for $P$, $T$ and $S$, the equations are to be integrated by approximation. See *Airy*, Lunar Theory, Art. 32, and the sequel: *Mrs Somerville*, Art. 687, and sequel: Mec. Cel. Seconde Partie, Liv. VII.

# CHAP. V.

## ON THE ATTRACTIONS OF BODIES.

In the preceding Chapter we have supposed bodies to be points, and attractive forces to tend to those points. If the bodies be of finite magnitude, and if attractive forces tend to each point of the mass, the result of all these forces will be a single force. In the present Chapter we determine the law of this force, corresponding to different laws of the elementary forces, and different figures of the attracting bodies.

### *Sect. I.* Geometrical Investigation of the Attractions of Bodies.

(Newton, Book I. Sect. xii.)

(Newton, Book I. Prop. lxx.)

90. Prop. *The force tending to each point of a body being supposed to vary inversely as the square of the distance, the force exerted by a spherical surface on a point within it will be nothing.*

Fig. 164. Let $HIKL$ be the spherical surface, $P$ the point attracted. Through $P$ draw $HK$, $IL$, cutting off small arcs $HI$, $KL$; the triangular figures $PIH$, $PKL$ will be similar, and the arcs $HI$, $KL$ will be proportional to $HP$, $LP$, (*Introd.* Lemma vi.) And if any number of such lines be drawn through $P$, forming a double cone or pyramid of very small angle, the small portions of the spherical surface thus intercepted will be as the squares of $HP$, $LP$. Now, these two portions of the spherical surface, ultimately, when they become indefinitely small, exert forces which are as the portion of the surface directly, and as the square of the distance inversely: and therefore their forces are equal; and being in opposite

directions they will balance each other. And in like manner the attractions of any two other portions of the spherical surface thus opposite will balance each other: and the whole attraction will be nothing.

Cor. 1. The attraction of a homogeneous hollow sphere on a point within it is nothing.

For the hollow sphere may be conceived to be made up of indefinitely thin concentric spherical surfaces, and each of these surfaces exerts no action on the point; therefore, the whole hollow sphere exerts no action upon it.

Cor. 2. Hence, the attraction of a hollow sphere upon a point in contact with its interior surface is nothing.

(Newton, Book I. Prop. LXXI.)

91. Prop. *On the same supposition, the force exerted by a spherical surface on a point without it, will tend to the centre of the sphere, and will be inversely as the square of the distance of the attracted point from the centre.*

Fig. 165. Let $AHKB$, $ahkb$, be two equal spherical surfaces, of which the centres are $S$, $s$; and in the diameters $BA$, $ba$ produced, let $P$, $p$ be the attracted points.

Let $PHK$, $PIL$, $phk$, $pil$, be drawn, in planes passing through the diameters $AB$, $ab$; and so that the arcs $HK$, $hk$ are equal, and $IL$, $il$. Draw perpendiculars $SD$, $sd$, which will also be equal; and perpendiculars $SE$, $se$, which will in like manner be equal. Let $SD$, $sd$ meet $PL$, $pl$ in $F$, $f$. Draw $IR$, $ir$ perpendicular to $PK$, $pk$; $IQ$, $iq$ to $PB$, $pb$.

Let the angles $DPE$, $dpe$ become indefinitely small; and since $DS - ES$ is always equal to $ds - es$; and that $DF$ is ultimately equal to $DS - ES$, and $df$ to $ds - es$; $DF$, $df$ are ultimately equal. We then have

$$\frac{PI}{PF} = \frac{RI}{DF}; \quad \frac{pf}{pi} = \frac{df}{ri}; \quad \text{whence } \frac{PI \cdot pf}{PF \cdot pi} = \frac{RI}{ri} = \frac{IH}{ih},$$

because the triangle $IRH$ is ultimately similar to $irh$, each being similar to $HDS$ or $hds$.

Again, $\frac{PI}{PS} = \frac{IQ}{SE}$; $\frac{ps}{pi} = \frac{se}{iq}$; whence $\frac{PI \cdot ps}{PS \cdot pi} = \frac{IQ}{iq}$, because $SE = se$. Hence, multiplying,

$$\frac{PI^2 \cdot pf \cdot ps}{pi^2 \cdot PF \cdot PS} = \frac{IH \cdot IQ}{ih \cdot iq} = \frac{\text{annulus described by } IH}{\text{annulus described by } ih};$$

the annulus in each case being supposed to be described by the revolution of the figure round the axis $AB$. For on this supposition the surface will, ultimately, be as the line $IH$, or $ih$, multiplied into the length of its path, that is, into the circle of which the radius is $IQ$ or $iq$; and these circles are as $IQ$, $iq$.

Each particle of the annulus described by $IH$, attracts $P$ directly with a force which is ultimately as $\frac{\text{particle}}{PI^2}$: and this force being resolved in the direction $PS$, gives a force which is as $\frac{\text{particle}}{PI^2} \cdot \frac{PQ}{PI}$, or as $\frac{\text{particle}}{PI^2} \cdot \frac{PF}{PS}$. Hence, the whole attraction of the annulus in the direction $PS$, is as

$$\frac{\text{annulus}}{PI^2} \cdot \frac{PF}{PS}.$$

And, therefore, comparing the attraction, in the direction $PS$, of the two annuli described by $IH$ and $ih$, and comparing the annuli by the above equation, we have

$$\text{att. of ann. } IH : \text{att. of ann. } ih :: pf \cdot ps \cdot \frac{PF}{PS} : PF \cdot PS \cdot \frac{pf}{ps}$$

$$:: \frac{1}{PS^2} : \frac{1}{ps^2}.$$

By similar reasoning, it appears that the attractions of the annuli $KL$, $kl$ in the direction $PS$, are in the same proportion.

And by taking $sd$, $se$, &c. equal to $SD$, $SE$, &c. the whole of the two spherical surfaces are thus divided into cor-

responding portions; and for each of those portions it appears that the action upon the particle $P$ tending to $S$, is inversely as the squares of the distances $SP$, $sp$. Also, the lateral forces perpendicular to the direction $SP$ are equal, and opposite in opposite parts of the sphere, and destroy each other. Therefore the whole actions of the two equal spherical surfaces tend to their centres, and are inversely as the squares of the distances: and therefore the attraction of the same spherical surface at different distances follows this law.

(NEWTON, Book I. Prop. LXXII.)

92. PROP. *On the same supposition, if two homogeneous spheres of the same density attract two points; and if the ratio of the radius of the sphere to the distance of the attracted point be the same in both; the action upon the attracted point will be proportional to the radii of the spheres.*

Let the spheres be separated into particles similar and similarly situated with regard to the attracted points. The direct attractions to similar particles in the two spheres will be as the particles directly, and the squares of the distances inversely; that is, as the cubes of the radii directly, and the squares of the radii inversely: or directly as the radii. And the resolved actions towards the centre of the sphere will be in the same proportion as the direct actions, by the similarity of the figures; that is, as the radii. Hence, the whole action in the two cases, being the sum of the actions of all the particles, will also be as the radii of the spheres.

COR. 1. The attractions of spheres of the same density upon points at their surface, are as the radii of the spheres.

COR. 2. The same reasoning would apply to any two similar solids of equal density, which attract points similarly situated with regard to the solids; the attractive forces tending to each point of the solids being supposed to vary inversely as the squares of the distances.

COR. 3. Hence, if two such similar spheroids, of equal density, attract points similarly situate on their surfaces, the

forces will be as the distances of the points from the centre of the spheroids.

(NEWTON, Book I. Prop. LXXIII.)

93. PROP. *On the same supposition, the attraction of a homogeneous sphere upon a point any where within it, is directly as the distance from the centre.*

Let a spherical surface, concentric with the sphere, pass through the attracted point. The hollow sphere outside this surface exerts no attraction on this point, by Cor. 2, Art. 90; and the attraction of the sphere within this surface is as the radius, by Cor. 1, Art. 92.

(NEWTON, Book I. Prop. LXXIV.)

94. PROP. *On the same supposition, the attraction of a homogeneous sphere upon a point without it, is inversely as the square of the attracted point from the centre of the sphere.*

The sphere may be conceived to be made up of concentric spherical shells, which ultimately may be considered as spherical surfaces; and the attraction of each such surface will be, by Art. 91, inversely as the square of the distance of the attracted point from the centre of the sphere. Therefore, the attraction of the whole sphere will be in this proportion.

COR. 1. Hence, at equal distances from the centres of homogeneous spheres, the attractions are as the spheres. Let $\alpha$, $\beta$ be the radii of the two spheres $A$, $B$; $a$ any distance. Then, by Art. 92,

$$\text{attr. of } A \text{ at dist. } a : \text{attr. of } B \text{ at dist. } \frac{\beta a}{\alpha} :: \alpha : \beta,$$

$$\text{attr. of } B \text{ at dist. } \frac{\beta a}{\alpha} : \text{attr. of } B \text{ at dist. } a :: a^2 : \frac{\beta^2 a^2}{\alpha^2}.$$

Hence,

$$\text{attr. of } A \text{ at dist. } a : \text{attr. of } B \text{ at dist. } a :: \alpha^3 : \beta^3 :: A : B.$$

Cor. 2. Hence, at any distance $b$ from $B$,

$$\text{attr. of } A \text{ at dist. } a : \text{attr. of } B \text{ at dist. } b :: \frac{A}{a^2} : \frac{B}{b^2}.$$

(Newton, Book I. Prop. lxxv.)

95. Prop. *On the same supposition, the attracting sphere attracts another homogeneous sphere with a force which is inversely as the square of the distance of the centres of the spheres.*

Let $A$ be the attracting, $B$ the attracted sphere.

The force of the sphere $A$ on any point, varies inversely as the square of the distance of the attracted point from its centre, and therefore is the same as if the attracted point were drawn by a single particle placed at the centre of the sphere $A$, and attracting inversely as the square of the distance of the points. And this is true of every point of the sphere $B$; and the effect is therefore the same as if each particle of the sphere $B$ attracted the centre of the sphere $A$, with forces of the same intensity. But in this case, by Art. 91, the centre of the sphere $A$ would be drawn towards the centre of $B$ by a force, which is inversely as the square of the distances of these centres; and by the same force the centre of $B$ would be drawn towards the centre of $A$. And therefore the effect of the attraction of $A$ on $B$, is to produce a force varying inversely as the square of the distance of the centres of these spheres.

Cor. 1. The attraction on a sphere $A$ by different homogeneous spheres, $B$, $C$, at distances $b$, $c$, is as $\frac{B}{b^2}$, $\frac{C}{c^2}$.

Cor. 2. If $A$ attract $B$ as well as $B$ attract $A$, the result will be the same. For each attraction will produce equal forces of action and reaction, by which the bodies are drawn together; and therefore, on the whole, the force will be doubled, and the law will remain the same.

(Newton, Book I. Prop. lxxvi.)

96. Prop. *On the same suppositions, if two spheres be of densities variable in proceeding from the centre to the circum-*

*ference, but constant at a constant distance from the centre, the whole force with which one sphere attracts the other will be inversely as the square of the distance of their centres.*

Let the spheres $A$, $B$, be divided into concentric shells; the density in each shell is uniform by the supposition. And hence each shell of $B$ exerts upon each point of $A$, a force which varies inversely as the distance of the point from the centre of $B$: and therefore the whole sphere $B$ exerts upon each point of $A$ a force, which varies inversely as the distance from the centre of $B$. And this force, *ceteris paribus*, will be as the density of the attracted point of $A$: and if a shell of $A$ be taken, the density in this shell will be constant; and therefore the result of all the forces acting on this shell, and tending to $B$'s centre, will be inversely as the square of the distance of $A$'s centre from $B$'s centre. And this is true of all the shells of which $A$ is composed: therefore the whole attraction of $A$ to $B$'s centre is in the same proportion.

Cor. 1. Hence, if $A$, $B$, $C$ be *similar* spheres attracting each other, the attractions of $A$ to $B$ and $C$, at equal distances, will be as $B$ to $C$:

Cor. 2. And at unequal distances $b$, $c$, as $\frac{B}{b^2}$ to $\frac{C}{c^2}$.

Cor. 3. The above attractions are the *accelerating forces* on $A$, or forces which are measured by the velocity produced or added in a given time. But the *moving forces* on $A$, or forces measured by the momentum produced, are at equal distances as $A . B$ and $A . C$.

The *pressure* which produces motion is known, by the laws of motion, to be as the momentum produced in a given time. And this pressure is the *weight* of one body towards another, or the force requisite to sustain it so that motion may be prevented. Hence, the weight of a sphere $A$ towards a sphere $B$ at a given distance, is as $A . B$:

Cor. 4. And at any distance $b$, of their centres, it is as $\frac{A . B}{b^2}$.

Cor. 5. The same is true when the attraction is mutual, as in Cor. 2, Art. 95. The attraction will be doubled, and the law will remain the same.

Cor. 6. If the spheres revolve upon their axes, the same law of attraction will remain unaltered.

(Newton, Prop. xci. Cor. 3.)

97. Prop. *The force tending to each point of a body being supposed to vary inversely as the square of the distance, a spheroidal shell of indefinitely small thickness, bounded by similar and concentric spheroidal surfaces, will exert no attraction upon a point situate within it.*

Fig. 166. Let $ADFE$, $HKLI$ be two similar and spheroidal surfaces, of which the common centre is $S$. Draw two lines, $DE$, $FG$, through $P$, the attracted point, meeting the surfaces in $D$, $E$, $H$, $I$ and $F$, $G$, $K$, $L$. The lines $DH$, $EI$ will be equal; for if we bisect $DE$ in $M$, $M$ will also be the bisection of $HI$, by the similarity of the figures: in like manner, $FK$ is equal to $GL$.

Let lines drawn through $P$ form a double pyramid or cone of small angle. The portion of the spherical shell $DHKF$, will ultimately be equal to $DHfk$; $Df$ and $Hk$ being planes perpendicular to $PD$. In like manner, $EILG$ will ultimately be equal to $EIlg$, similarly determined.

And since the thicknesses $DH$, $EI$ of these particles of the shell are equal, the particles will be as the planes $Df$, $Eg$; that is, as $DP^2$, $EP^2$, by similar figures. And the attractions of these particles being as the particles directly, and as the squares of the distances inversely, are equal. And they are opposite; therefore they destroy each other. In like manner, the whole spheroidal shell may be resolved into pairs of particles, which destroy each other's effect, and the resulting attraction is nothing.

Cor. 1. The attraction of a *concentrically homogeneous* hollow spheroid, bounded by similar and concentric spheroidal surfaces upon a point within, is nothing.

For the hollow spheroid may be conceived to be made up of indefinitely thin shells bounded by similar concentric spheroidal surfaces; and each of these shells exerts no attraction on the point, by the Proposition. Therefore the whole hollow spheroid exerts no attraction.

COR. 2. Hence, the attraction of such a hollow spheroid upon a point in contact with its interior surface is nothing.

COR. 3. If a point be placed any where within a solid homogeneous spheroid, the attraction will be as the distance from the centre. For let a similar and concentric spheroidal surface pass through the attracted point. The hollow spheroid outside this surface exerts no attraction on this point by Cor. 2. and the attraction of the spheroid inside the surface is as the distance from the centre as in Art. 95.

(NEWTON, Book I. Prop. LXXXVIII.)

98. PROP. *The forces tending to each point of a body being supposed to vary directly as the distance, a body of any figure will exert upon any point a force which tends to the centre of gravity of the body, and varies in the distance of the attracted point from that centre.*

Fig. 167. Let $Z$ be the attracted point; and let $A$, $B$ be two particles of the attracting body. Then the forces to $A$ and $B$ will be as $A.AZ$ and $B.BZ$: and if $G$ be the centre of gravity of $A$, $B$, the force $A.AZ$ may be resolved into two forces, in the directions respectively of the lines $ZG$, $GA$ which will be as $A.ZG$ and $A.GA$. In like manner the force $B.BZ$ may be resolved into $B.ZG$ and $B.GB$. And by the property of the centre of gravity $A.GA = B.GB$; therefore the two forces in $GA$, $GB$ destroy each other; and the point $Z$ is attracted towards $G$ with a force $(A + B) ZG$.

In like manner if we take a third particle $C$, and compound its attraction $(C.CZ)$ with the force $(A + B)\ GZ$, tending to $G$, we shall find a third force tending to the common centre of $A$, $B$, $C$, and equal to $A + B + C$ multiplied

into the distance of $Z$ from this centre. And this force will be the whole effect of $A$, $B$, $C$. And the effect will manifestly be the same as if $A$, $B$, $C$ were collected in their common centre of gravity.

In like manner if we take an indefinite number of such particles, that is, a solid body, the whole effect of their attraction will be the same as if they were all collected in their centre of gravity: therefore the force will tend to this centre, and will vary as the distance from it.

## *Sect. II.* Analytical Investigation of Attractions.

99. Prop. *A spherical shell of indefinitely small thickness, being composed of particles attracting according to a given law; to find the attraction on any point.*

Let $S$, fig. 168, be the centre of the spherical shell, $SE$, its radius $= a$; $EF$, its thickness $= \delta a$; $P$ the point attracted; $PS = r$, $PF = f$, $F$ being any particle; and let $PSE = \theta$.

Suppose two planes $FSP$, $GSP$, passing through $SP$, to make with each other a small angle $FDG = \delta\phi$, $FDG$ being a plane perpendicular to $PD$. Then,

$$FG = DF \,.\, \delta\phi = a \sin.\theta \,.\, \delta\phi.$$

And if we suppose $ESe = \delta\theta$, $Ee = a\delta\theta$; and the solid content of the small portion of the shell $EFGe$ will be

$$\delta\phi \,.\, \delta a \,.\, \delta\theta \,.\, a^2 \sin.\theta.$$

Now when this portion is indefinitely small, its attraction on $P$ may be considered as that of a single particle at the distance $f$. Let $\phi(f)$ be the function of $f$ expressing the law of attraction; then the attraction of the elementary solid $Ge$ on $P$ will be $\delta\phi \,.\, \delta a \,.\, \delta\theta \,.\, a^2 \sin.\theta \,.\, \phi(f)$. To reduce this to the direction $PS$, we must multiply it by

$$\frac{PD}{PF}, \text{ or } \frac{r - a\cos.\theta}{f};$$

hence, the attraction towards $S$, is

$$\delta\phi \,.\, \delta a \,.\, \delta\theta \,.\, a^2 \sin.\theta \,.\, \phi(f) \,.\, \frac{r - a\cos.\theta}{f};$$

and the quantity which multiplies $\delta\theta$ being considered as a differential coefficient, and integrated from $A$ to $B$, we shall have the attraction of the slice $AEB$ towards $S$

$$= \delta\phi \,.\, \delta a \int_\theta a^2 \sin.\theta \,.\, \phi(f) \,.\, \frac{r - a\cos.\theta}{f}.$$

The attraction, by altering the angle $\delta\phi$, manifestly varies in the same ratio; hence, for the whole shell, we must put $2\pi$ for $\delta\phi$, and we have for the whole attraction

$$2\pi a^2 \delta a \int_\theta \sin.\theta \,.\, \phi(f) \frac{r - a\cos.\theta}{f} = A, \text{ suppose.}$$

$$\text{Since } f^2 = r^2 - 2ra\cos.\theta + a^2, \quad \frac{r - a\cos.\theta}{f} = \left(\frac{df}{dr}\right),$$

the differential coefficient when $r$ alone is supposed to vary.

$$\text{Hence, } A = 2\pi a^2 . \,\delta a \int_\theta \sin.\theta \,.\, \phi(f) \,.\, \left(\frac{df}{dr}\right).$$

Now let $\int_f \phi(f) = \phi_1(f)$, and we have, since $\phi(f)$ is the differential coefficient of $\phi_1(f)$,

$$\left(\frac{d\phi_1(f)}{dr}\right) = \phi(f) \,.\, \left(\frac{df}{dr}\right).$$

Hence, if we take $B = 2\pi a^2 a \int_\theta \sin.\theta \phi_1(f)$, we shall have

$$\frac{dB}{dr} = 2\pi a^2 . \,\delta a \int_\theta \sin.\theta \,.\, \left(\frac{d\phi_1(f)}{dr}\right)$$

$$= 2\pi a^2 . \,\delta a \int_\theta \sin.\theta \,.\, \phi(f) \,.\, \left(\frac{df}{dr}\right) = A,$$

for since the variations of $\theta$ and of $r$ are independent, it makes no difference whether we perform the differentiations before or after integration*.

Now, since $f^2 = r^2 - 2ra\cos.\theta + a^2$, we have

$$f\frac{df}{d\theta} = ra\sin.\theta;$$

and $\sin.\theta = \dfrac{f}{ra}\cdot\dfrac{df}{d\theta}$;

hence, $B = \dfrac{2\pi a\delta a}{r}\int_\theta f\phi_1(f)\,\dfrac{df}{\delta\theta} = \dfrac{2\pi a\delta a}{r}\int_f f\cdot\phi_1(f)$.

The integral is to be taken from $\theta = 0$, to $\theta = \pi$; that is, from $f = r - a$, to $f = r + a$. If $\int_f f\cdot\phi_1(f) = \psi(f)$, we have for the whole figure,

$$B = \frac{2\pi a\delta a}{r}\{\psi(r+a) - \psi(r-a)\}.$$

And the attraction $= A = \dfrac{dB}{dr}$ is thus known.

For a point within the shell, the process will be the same, except that the integral must be taken between the limits $a + r$, and $a - r$.

Ex. 1. Let the force of each particle vary inversely as the square of the distance.

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* If $F$ be a function of $r$ and $\theta$, and $B = \int_\theta F$,

$$\frac{dB}{dr} = \frac{d\int_\theta F}{dr}.$$

But $\dfrac{dB}{d\theta} = F$, $\dfrac{dF}{dr} = \dfrac{d^2B}{d\theta\,dr} = \dfrac{d^2B}{dr\,d\theta}$.

(Lacroix, *Elem. Treat.* 122.) Hence, integrating to $\theta$,

since $\dfrac{dF}{dr} = \dfrac{d^2B}{dr\,d\theta}$, $\int_\theta\dfrac{dF}{dr} = \dfrac{dB}{dr} = \dfrac{d\int_\theta F}{dr}$.

Here $\phi(f) = \frac{1}{f^2}$; $\phi_1(f) = \int_f \cdot \phi(f) = -\frac{1}{f}$;

$$\psi f = \int_f f\phi_1(f) = -f,$$

$$B = \frac{2\pi a\delta a}{r}\{(r-a)-(r+a)\} = -\frac{4\pi a^2 a}{r},$$

$$A = \frac{dB}{dr} = \frac{4\pi a^2\delta a}{r^2}.$$

The surface of the shell is $4\pi a^2$; and hence its mass is $4\pi a^2\delta a$, and the attraction is the same as if the mass were collected at the centre of the sphere.

Ex. 2. Let the force of each particle vary as any power of the distance.

Let $\phi(f) = f^n$, whence $\phi_1(f) = \frac{f^{n+1}}{n+1}$, $\psi(f) = \frac{f^{n+3}}{(n+1)(n+3)}$,

$$B = \frac{2\pi a\delta a}{(n+1)(n+3)r}\{r+a)^{n+3} - (r-a)^{n+3}\}$$

$$= \frac{4\pi a^2 a}{n+1}\{r^{n+1} + \frac{(n+2)(n+1)}{2.3}r^{n-1}a^2$$

$$+ \frac{(n+2)(n+1)n.(n-1)}{2.3.4.5}r^{n-3}a^4 + \&c.\}.$$

$$\text{And } A = \frac{dB}{dr} = 4\pi a^2\delta a\{r^n + \frac{(n+2)(n-1)}{2.3}r^{n-2}a^2$$

$$+ \frac{(n+2)n(n-1)(n-3)}{2.3.4.5}r^{n-4}a^4 + \&c.\}.$$

This series terminates, if $n$ be a whole positive number.

If $n=1$, or $n=-2$, that is, if the attraction varies directly as the distance, or inversely as the square of the distance, the terms after the first vanish; and the attraction is the same as if the mass were collected at the centre of the sphere.

Hence, if the particles exert a force which is as $mr + \frac{m'}{r^2}$, the whole force will be the same as if the mass were so collected; for we may suppose the shell to consist of particles which attract with forces $mr$, and of an equal number of others which attract with forces $\frac{m'}{r^2}$.

If $n = -1$, or $n = -3$, the integrations for $\psi(f)$ fail, and we must employ other methods.

Ex. 3. Let the force vary inversely as the cube of the distance.

$$\phi(f) = \frac{1}{f^3}, \quad \phi_1(f) = -\frac{1}{2f^2}, \quad \psi(f) = -\tfrac{1}{2}\,\mathrm{l}f.$$

$$B = \frac{\pi a \delta a}{r}\,\mathrm{l}\,\frac{r-a}{r+a},$$

$$A = \frac{dB}{dr} = \frac{\pi a \delta a}{r^2}\left\{\frac{2ar}{r^2 - a^2} - \mathrm{l}\,\frac{r-a}{r+a}\right\}.$$

Ex. 4. Let the force vary inversely as the distance,

$$A = \pi a \delta a \left\{\frac{2a}{r} + \left(1 - \frac{a^2}{r^2}\right)\mathrm{l}\,\frac{r+a}{r-a}\right\}.$$

Ex. 5. The force varying as any power of the distance; to find the attraction on a point within the shell.

As in Ex. 2, $\psi(f) = \dfrac{f^{n+3}}{(n+1)(n+3)}$,

$$B = \frac{2\pi a \delta a}{(n+1)(n+3)r}\left\{(a+r)^{n+3} - (a-r)^{n+3}\right\}$$

$$= \frac{4\pi a}{n+1}\left\{a^{n+3} + \frac{(n+2)(n+1)}{2 \cdot 3}a^{n+1}r^2\right.$$

$$\left. + \frac{(n+2)(n+1)n(n-1)}{2 \cdot 3 \cdot 4 \cdot 5}a^{n-1}r^4 + \&c.\right\},$$

$$\therefore A = \frac{dB}{dr} = 4\pi a\delta a \left\{ \frac{n+2}{3} a^{n+1} r \right.$$

$$\left. + \frac{2(n+2)n(n-1)}{3 \, . \, 4 \, . \, 5} a^{n-1} r^3 + \&c. \right\}.$$

If $n = -2$, or the force be inversely as the square of the distance, we have $A = 0$; the attractions in different directions counterbalance each other.

100. Prop. *To find the attraction of a sphere composed of particles attracting according to a given law.*

If in the last proposition we put $u$ for $a$, consider the coefficient of $\delta a$ or $\delta u$ as a differential coefficient, and integrate from $u = 0$, to $u = a$, we shall have the attraction of a solid sphere of radius $a$.

By this means, from the expression for $A$ in Ex. 2, we find for the attraction of a sphere

$$\frac{4\pi a^3}{3} \left\{ r^n + \frac{(n+2)(n-1)}{2} . \frac{3 r^{n-2} a^2}{5} + \&c. \right\}.$$

In the cases where the attraction of a shell is the same as if the matter were collected at the centre, the attraction of a sphere will also follow the same law. For the sphere may be supposed to be composed of concentric shells, each of which attracts as if it were collected at the centre, and therefore the whole sphere will attract as if all its parts were there collected.

101. Prop. *To find the attraction of a circle on a point in a line perpendicular to its plane, and passing through its centre.*

Let $BC$, fig. 169, be the circle; $P$, the attracted point; $SP = r$, $PE = f$, $SE$ any radius $= u$, and $SF$ a radius indefinitely nearly equal to this, so that $EF = \delta u$. Let a small angle $FSG = \delta \phi$, then the quadrilateral $EG = u \, . \, \delta\phi \, . \, \delta u$. And, if the law of the attraction be represented by $\phi(f)$,

the attraction of $EG$ is $\delta\phi \,.\, \delta u \,.\, u \,.\, \phi(f)$, which, resolved in the direction $PS$, becomes $\delta\phi \,.\, \delta u \,.\, u \,.\, \phi(f) \,.\, \frac{r}{f}$. And for the whole annulus whose breadth is $EF$, we must put $2\pi$ for $\delta\phi$; whence it becomes $2\pi u \delta u \,.\, \phi(f) \,.\, \frac{r}{f}$. Hence, the attraction of the whole circle

$$= 2\pi \textstyle\int_u u\phi(f) \frac{r}{f}; \quad \text{where } f = \sqrt{(r^2 + u^2)};$$

the integral being taken from $u = 0$, to $u = a$, the radius of the circle.

If $\phi(f) = f^n$,

$$\text{attraction} = 2\pi r \textstyle\int_u u (r^2 + u^2)^{\frac{n-1}{2}}$$

$$= \frac{4\pi r}{n+1} (r^2 + u^2)^{\frac{n+1}{2}} + \text{constant}$$

$$= \frac{4\pi r}{n+1} \left\{ (r^2 + a^2)^{\frac{n+1}{2}} - r^{n+1} \right\}.$$

Ex. 1. Let $n = -2$, or the force vary inversely as the square of the distance.

Here, attraction $= 4\pi \left\{ 1 - \dfrac{r}{\sqrt{(r^2 + a^2)}} \right\}$.

Ex. 2. Let the circle be infinite, and $n < -1$.

In this case $(r^2 + a^2)^{\frac{n+1}{2}}$ becomes 0, and we have, putting $-m$ for $n$,

$$\text{attraction} = \frac{4\pi}{(m-1) r^{m-2}}.$$

If $m = 2$, or the force vary inversely as the square of the distance, attraction $= 4\pi$, and is the same at all distances.

102. **Prop.** *To find the attraction of a solid of revolution on a point in the axis.*

We must here multiply the attraction of the circle, found in the last Proposition, by the thickness $\delta r$, for the attraction of a differential slice; and if we then put for $a$ its value in terms of $r$, and integrate the coefficient with respect to $r$, we have the attraction of the whole solid.

Ex. 1. The attraction of a cylinder on a point in its axis; fig. 170.

$$\text{attraction} = \frac{4\pi}{n+1} \int_r \left\{ r(r^2+a^2)^{\frac{n+1}{2}} - r^{n+2} \right\}$$

$$= \frac{4\pi}{n+1} \left\{ \frac{(r^2+a^2)^{\frac{n+3}{2}}}{n+3} - \frac{r^{n+3}}{n+3} + \text{constant} \right\}.$$

If $BSC$ and $B'S'C'$ be the two ends of the cylinder, and if $PS = b$, $PS' = b'$, $PC = c$, $PC' = c'$, we have

$$\text{attraction} = \frac{4\pi}{(n+1)(n+3)} \left\{ c'^{n+3} - c^{n+3} - (b'^{n+3} - b'^{n+3}) \right\}.$$

If the force vary inversely as the square of the distance, $n = -2$,

$$\text{attraction} = 4\pi \left\{ b' - b - (c' - c) \right\}.$$

Ex. 2. The attraction of an infinite solid bounded by planes.

From last Prop. Ex. 2, we have

$$\text{attraction} = \frac{4\pi}{m-1} \int_r \frac{1}{r^{m-2}} = \frac{4\pi}{(m-1)(m-3)} \left\{ \frac{1}{b^{m-3}} - \frac{1}{r^{m-3}} \right\},$$

where $b$ is the distance of the attracted point from the surface of the solid.

If $m = 2$, attraction $= 4\pi(r - b) = 4\pi \times$ thickness.

If $m = 3$, attraction $= 2\pi \, \mathrm{l} \dfrac{r}{b}$.

If $m > 3$, the attraction is finite, when $r$ is infinite, and we have

$$\text{attraction} = \frac{4\pi}{(m-1)(m-3)b^{m-3}}.$$

Ex. 3. The attraction of a cone on a point at the vertex.

In fig. 171, let $PS = r$, $ST = kr$, and putting $kr$ for $a$ in last Prop.

$$\text{attraction} = \frac{4\pi}{n+1} \int_r r^{n+2} \{(1+k^2)^{\frac{n+1}{2}} - 1\}$$

$$= \frac{4\pi r^{n+3}}{(n+1)(n+3)} \{\sqrt{(1+k^2)} - 1\}.$$

Where $r$ is to be made $= PA$ the axis.

123. Prop. *To find the attraction of a straight line upon a point at any distance from it.*

Let $BC$, fig. 172, be the attracting line, $P$ the point attracted: $PS$, perpendicular on $BC$, $= r$, $SE = u$, $PE = f$; and let the force of each particle be as $\phi(f)$. We may suppose $EF$, an indefinitely small portion, to be $\delta u$; and its attraction on $P$ will be $\phi(f)\,\delta u$; and the part resolved *perpendicular* to $BC$ will be $\dfrac{r}{f}\phi(f)\,\delta u$, where $f = \sqrt{(r^2 + u^2)}$. The coefficient is to be integrated from $u = 0$, to $u = a = SB$, for the attraction of $SB$; and the attraction of $SC$ is to be found in the same manner, and added to the former.

The attraction of $\delta u$ *parallel* to $SB$, will be $\dfrac{u}{f}\phi(f)\,\delta u$; which is to be integrated in the same manner as before; and the difference taken, of the parts belonging to $SB$ and to $SC$.

Ex. Let the force vary inversely as the square of the distance. Here $\phi(f)=\frac{1}{f^2}$;

$$\text{attraction in } PS=\int_u \frac{r}{(r^2+u^2)^{\frac{3}{2}}}=\frac{u}{r\sqrt{(r^2+u^2)}},$$

$$\text{attraction perpendicular to } PS=\int_u \frac{u}{(r^2+u^2)^{\frac{3}{2}}}=\frac{1}{r}-\frac{1}{\sqrt{(r^2+u^2)}}.$$

And this is to be taken for $SB$ and for $SC$, and the results combined.

For the attractions of spheroids and ellipsoids, see *Airy*, Fig. Earth, Articles 8 and 41.

## *Sect. III.* The Figure of the Earth.

(Newton, Book III. Prop. xviii.)

104. Prop. *The equatoreal diameters of the planets are greater than their axes of revolution.*

The planets are supposed to be homogeneous masses, of which the parts are in equilibrium.

If a mass of homogeneous fluid be at rest, and acted upon only by its own attraction, it will be in equilibrium when it is in the form of a sphere. For the attraction will in this case be equal on all points of the surface, and every point will be similarly situated with regard to the surrounding parts, and can therefore have no reason to move in one direction or in another: all the points will therefore remain at rest by their mutual action.

But if a mass of homogeneous fluid revolve on an axis, it will no longer be in equilibrium in a spherical form. For in consequence of the motion of revolution, all the parts will have a tendency to recede from the axis, and will not keep their relative places except they be retained by the pressures and attractions of the other parts. And if the largest section perpendicular to the axis of revolution be called *the equator*, the parts about the equator, being farther from the axis, and moving quicker than the rest, will tend to recede from the

axis more than the others, and will recede, except they be restrained by some opposing force, which acts upon them more than upon the other parts; and in the spherical form there will be no force, thus acting unequally on different parts; therefore there cannot be equilibrium.

If the mass assume the form of an oblate spheroid of revolution, having for its axis the axis about which the fluid revolves, there is an unequal attraction upon the parts about the pole and about the equator: also the form and quantity of the fluid is different in those parts of the figure: therefore there may be some adjustment of the form of the spheroid by which equilibrium may be preserved, when the fluid revolves: the tendency of the equatoreal parts to recede from the axis being counteracted by the greater dimensions of those parts and the greater effect of the attraction upon them.

In what follows we suppose the attractive force of each particle to vary inversely as the square of the distance.

An *oblate* spheroid has its equatoreal diameter greater than the polar, a *prolate* spheroid has the polar greater than the equatoreal.

105. PROP. *In a homogeneous symmetrical oblate spheroid very little different from a sphere, the ratio of the semidiameters being* $1+\epsilon : 1$; *the attraction of the inscribed sphere on a point at its surface, of the spheroid on a point at its pole, and of the same spheroid on a point at its equator, are respectively as* $1$, $1+x$, $1+\epsilon-\frac{x}{2}$.

If we suppose, in the oblate spheroid, a sphere to be inscribed, having the same axis as the spheroid, the part of the spheroid outside the sphere will be a stratum thickest at the equator and thinning off symmetrically towards the two poles, which stratum is called a *meniscus*. The attraction on the point at the pole of the spheroid is equal to the attraction of the inscribed sphere, together with that of the meniscus. The attraction of the sphere on a point at the surface, the density being given, is as the radius. Let $b$ be the polar semidiameter,

and $(1+\epsilon)b$ the equatoreal: the attraction of the inscribed sphere may be represented by $b$, and if $bx$ represent the attraction of the meniscus, $b(1+x)$ will be the attraction of the oblate spheroid on a point at its pole.

If instead of adding the meniscus to the inscribed sphere, we take away an equal meniscus, we obtain a prolate spheroid of which the axes are $b$ and $(1-\epsilon)b$; and of which the attraction on a point at the pole is the attraction of the sphere *minus* the attraction of the meniscus. And the attraction of this meniscus is manifestly, when $\epsilon$ is small, ultimately equal to that of the former; therefore $b(1-x)$ represents the attraction of the prolate spheroid on a point at the pole.

In like manner if we circumscribe a sphere about either spheroid, the attraction of this sphere on a point at the surface will be $b(1+\epsilon)$; and if we have a prolate spheroid of which the semidiameters are $b(1+\epsilon)$ and $b$, (its pole being supposed to lie in the equator of the oblate spheroid,) the attraction of this prolate spheroid on a point at its pole will be $b(1+\epsilon)(1-x)$.

Now if we compare this prolate spheroid, the oblate spheroid, and the circumscribing sphere, it will appear that the first differs from the second, and the second from the third by semimeniscuses which are of the same kind, and ultimately equal: and the attraction of these semimeniscuses will be equal. And the attraction of both = attraction of circumscribing sphere − attraction of prolate spheroid on pole

$$= b(1+\epsilon) - b(1+\epsilon)(1-x) = bx(1+\epsilon).$$

Therefore the attraction of one semimeniscus $= \frac{1}{2}bx(1+\epsilon)$. Hence the attraction of the oblate sphere on a point at its equator = attraction of circumscribing sphere − attraction of semimeniscus $= b(1+\epsilon) - \frac{1}{2}bx(1+\epsilon)$; or, since $x$ and $\epsilon$ are small, so that their product may be neglected, attraction of oblate spheroid on a point in its equator $= b\left(1+\epsilon-\frac{x}{2}\right)$.

Cor. 1. By the application of Art. 103, or by other methods (*Airy*, Figure of the Earth, Art. 8.) it may be shewn that the attraction of the meniscus on a point at its pole is $\frac{4}{5}b\epsilon$. Hence $x = \frac{4\epsilon}{5}$ and $1 + \epsilon - \frac{x}{2} = 1 + \frac{3\epsilon}{5}$. The attractions of the oblate spheroid on a point at the pole and at the equator respectively are $b\left(1 + \frac{4\epsilon}{5}\right)$ and $b\left(1 + \frac{3\epsilon}{5}\right)$.

Cor. 2. If the spheroid be divided by concentric and similar surfaces, the attraction of each of the hollow ellipsoids on a point with it is nothing. Hence the attraction on points within the ellipsoid, is their attraction to the ellipsoid in the surface of which they are. And hence the attraction at different points in the same semidiameters, will be as the distance from the centre. Art. 92. Cor. 2.

Cor. 3. Hence if we take a concentric spheroid of which the polar and equatoreal semidiameters are $r$ and $r(1 + \epsilon)$, the attractions at the extremities of these semidiameters will be

$$r\left(1 + \frac{4\epsilon}{5}\right) \text{ and } r\left(1 + \frac{3\epsilon}{5}\right).$$

If $T$ be the time of revolution of the spheroid, then at the distance $r$ from the axis the velocity is $\frac{2\pi r}{T}$, and the centrifugal force is $\frac{4\pi^2 r}{T^2}$, (Art. 27); and is as $r$.

In the case of the earth, $T = 23^h\ 56' = 86164''$, and at the equator $r = 3985$ miles nearly; and the velocity which expresses the force of gravity at the surface is $32\frac{1}{6}$ feet a second; hence, nearly,

$$\text{centrifugal force : gravity} :: \frac{4\pi^2 \times 3985 \times 5280}{(86164)^2} : 32\tfrac{1}{6} :: \frac{1}{289} : 1.$$

106. Prop. *To find the proportion of the axis of a planet to its equatoreal diameter by Newton's method.*

(Newton, Book III. Prop. xix.)

Let there be a hypothetical spheroid of which the axes are as 101 to 100. Then $\epsilon$ is here $\frac{1}{100}$, and the polar and equatoreal attractions are as $1 + \frac{4}{500}$ and $1 + \frac{3}{500}$, or as 1,008 and 1,006. And the attractions at proportional distances from the centre in the polar and equatoreal semidiameters are in this same proportion. If these two semidiameters be divided into the same number of proportional parts, the parts will each be in the proportion 100 to 101, or 1 to 1,01. Hence the weight of the columns which lie in the direction of the polar and equatoreal diameters will be as $1 \times 1,008$ to $1,01 \times 1,006$, or as 1,008 to 1,016, or as 1 to 1,008 nearly. Hence if the rotatory velocity be such as to take off the fraction ,008 from the superficial equatoreal weight, the equilibrium of the spheroid will subsist; for the centrifugal force will then take off the same fraction of the weight from each particle in the plane from the equator. Therefore the equilibrium will subsist if the centrifugal force be ,008 of the equatoreal gravity: that is, if the centrifugal force be $\frac{8}{1008}$, or $\frac{1}{126}$ of the whole equatoreal force.

The centrifugal force, in the case of equilibrium, must be as the excess of the equatoreal semidiameter: for the centrifugal force must balance the attraction of the meniscus, and the excess of the equatoreal column in length; and these are each as the excess of the equatoreal semidiameter. Hence the excess of the equatoreal semidiameter for any other centrifugal force will be found by a proportion.

Now the centrifugal force in the case of the earth is $\frac{1}{289}$ of the equatoreal gravity: and in the hypothetical spheroid it is $\frac{1}{126}$; hence $\frac{1}{126} : \frac{1}{289} :: \frac{1}{100} : \frac{1}{229}$, and the ratio of the diameters of the earth is as 229 to 230.

The radius of the earth is about 3985 miles; hence the excess of the equatoreal semidiameter is about 14 miles.

Cor. 1. If $c$ be the ratio of the centrifugal force to gravity at the equator, the proportional parts of the polar and equatoreal axis, multiplied into the forces which act upon them will be

$$1 \times \left(1 + \frac{4\epsilon}{5}\right) \text{ and } (1+\epsilon) \times (1 + \frac{3\epsilon}{5} - c);$$

$$\text{or, } 1 + \frac{4\epsilon}{5} \text{ and } 1 + \frac{8\epsilon}{5} - c, \text{ omitting } \epsilon^2,$$

and these must be equal, whence $c = \dfrac{4\epsilon}{5}$.

Hence $c$ is known from $\epsilon$, and conversely.

Cor. 2. If $T$ be the time of revolution, the centrifugal force at distance $r = \dfrac{4\pi^2 r}{T^2}$. Hence $c = \dfrac{4\pi^2 r}{T^2}$: and $\epsilon$ is inversely as $T^2$.

If the density vary in different cases, the attraction will vary in the same proportion; the centrifugal force will bear to the attraction a smaller proportion; and the excess of the equatoreal diameter must be diminished in the same proportion in which the density is increased.

The earth revolves in $23^h\ 56'$, Jupiter in $9^h\ 56'$. Hence the ellipticity of Jupiter would be to that of the earth as the squares of these numbers, that is, as 29 to 5. Also the density of Jupiter is to that of the earth as $94\frac{1}{2}$ to 400 (taking Newton's numbers:) hence the difference of Jupiter's semidiameters will be to his smaller semidiameter as $\dfrac{29}{5} \times \dfrac{400}{94\frac{1}{2}} \times \dfrac{1}{229}$ or as 1 to $9\frac{1}{3}$ nearly.

It appears by mathematical investigation (*Airy*, Fig. Earth, 27.) that when the figure of the planet is an elliptical spheroid, there is an equilibrium among the pressures of all parts, as well as between those of the equatoreal and polar canals.

COR. 3. The force of gravity at the equator is to that at the pole as $1 + \frac{3\epsilon}{5} - c$ to $1 + \frac{4\epsilon}{5}$, that is, as 1 to $1+\epsilon$, ultimately. And *the force of gravity increases in going from the equator to the pole;* and the increment is nearly as the square of the sine of the latitude. (*Airy*, Fig. Earth, 63.)

COR. 4. The direction of gravity is the direction of a normal to the surface of the spheroid; and two places on the same meridian differ a degree in latitude, which are so situated that their normals are inclined to each other one degree. Such normals of the elliptical spheroid will meet nearer the surface when the places are near the equator, and further from it when the places are near the pole. Hence the linear length of the *degrees of latitude* on the earth's surface will be *smaller nearer the equator and larger nearer the pole.* (*Airy*, Fig. Earth, 75.)

107. PROP. *The action of the Sun and Moon upon the protuberant parts of the elliptical spheroid will produce a Precession of the Equinoctial Points, and a Nutation of the Earth's Axis.*

(NEWTON, Book III. Prop. XXI.)

This appears by the reasoning of Art. 74. It may also be shewn by considering the effect of the attraction of the sun upon the protuberant meniscus of the spheroid. For the effect of this attraction, if the earth were at rest, would be to give to the spheroid a motion of rotation by which the axis would tend to become perpendicular to the line drawn from the sun to the earth's centre: and this motion of rotation, compounded with the diurnal rotation which the earth possesses, will produce a rotation about a new axis, as stated in Art. 75. And thus there will be a perpetual change of the axis, which when properly investigated is found to give rise to a precession of the equinoctial point. (*Airy*, Precession, 16.)

Also the attraction of the moon will produce a similar effect; and this, depending on the position of the moon's orbit, will periodically alter the position of the earth's axis with regard to the ecliptic, and produce a *lunar nutation.*

In these effects the moving force is the attraction of the sun and moon upon the protuberant meniscus of the spheroid: and the quantity of matter moved is the whole mass of the earth. The moving force is small compared with the inertia, because the thickness of the meniscus is small compared with its diameter, and its diameter small compared with the distance of the attracting bodies. Hence the effects just described will be very small.

## *Sect. IV.* The Tides.

108. Prop. *A Flux and Reflux of the ocean will be produced by the attraction of the Sun and Moon.*

(Newton, Book III. Prop. xxiv.)

It has been shown in Art. 73, that if a spheroid be covered with water, and revolve on its axis *in consequentia*, the waters will be elevated about the octants after syzygy. Hence, there will be high water about 3 hours after the attracting luminary is on the meridian. The moon attracts the waters more than the sun does; hence, the time of high water will be about 3 hours after the moon passes the meridian. When the sun and moon are in syzygy, the high water due to the one is added to the high water due to the other, and the tides are greatest. When the sun and moon are in quadratures, the sun produces the highest water at the place where the moon produces the lowest; and the apparent tide is only the difference of the effects. At other times, the tide, which by the action of the moon would be 3 hours after her southing, and by the action of the sun 3 hours after noon, will be at some intermediate time, but nearer to the former time. When the moon is going from syzygy to quadrature, the 3 o'clock of solar time precedes the 3 hours of lunar reckoning, and high water is *before* 3 hours of lunar reckoning; and the interval will be greatest a little after the moon's octants: as the moon passes from quadratures to syzygies, the time of high water will in like manner fall *behind* the third hour of lunar reckoning.

Such would be the case in the open sea: but the effects of shallow places, narrow seas, and continents, may retard the time by several hours.

If the globe of the earth were covered with water, the effect of the moon would be to convert this globe into a prolate spheroid, the pole of which would follow the moon's motion at an interval of about 3 hours. And the protuberance of this spheroid may be considered as composed of two meniscuses or waves, one under the moon or at some hours after, and one on the opposite side of the earth. In like manner, the sun would produce two such waves; and the joint action of the two luminaries would produce a spheroid which also may be considered as compounded of two such waves. And all places by their diurnal revolution are carried through both these tidal waves, and thus have high water twice in 24 hours. And the height of high water will vary according to the relative position of the sun and moon, and will go through a cycle of magnitudes twice a month. At or near new and full moon, there will be the highest or *spring* tides; at or near the first and third quarters of the moon will be the lowest or *neap* tides.

109. If we suppose, in a prolate spheroid of small excentricity, $\phi$ to represent the angle made with the axis by any line $r$ drawn from the centre to the surface, we shall have

$$r = a + h \cos.^2 \phi \text{ nearly,}$$

where $a$ is the mean radius, and $h$ is the elevation of the pole of the spheroid above its equator. And if the pole of the spheroid have a declination $\frac{\pi}{2} - \delta$ from the equator of the earth; and if $\theta$ be the difference of the longitudes of any place on the equator and of this pole, (that is, the hour angle of the luminary which is on the same meridian as the pole of the spheroid), we have $\cos. \phi = \cos. \theta \sin. \delta$,

$$r = a + h \sin.^2 \delta \cos.^2 \theta = a + \frac{h}{2} \sin.^2 \delta (\cos. 2\theta + 1)$$

$$= a + \frac{h}{2} \sin.^2 \delta + \frac{h}{2} \sin.^2 \delta \cos. 2\theta.$$

In the course of a diurnal revolution of the earth, $\theta$ changes through a circumference, but $\delta$ remains constant; therefore $a + \frac{h}{2}\sin.^2\delta$ is the mean value of $r$, and $\frac{h}{2}\sin.^2\delta\cos.2\theta$ is the elevation or depression of the surface above the mean, that is, the tide, in any state, at any moment of time.

In like manner, if $h'$ be the elevation of the pole of the spheroid produced by another luminary, $\theta'$ its hour angle from the meridian of the same place, $\delta'$ its angular distance from the pole of the earth,

$$\frac{h'}{2}\sin.^2\delta'\cos.2\theta'$$

will be the elevation or depression produced by this spheroid at the place.

When the excentricity of the two spheroids is very small, (as is the case in the tidal spheroids,) the elevation or depression resulting from the joint action of the luminaries, will be the sum of the separate elevations or depressions; that is, if $\frac{1}{2}H$ be the whole elevation above the mean

$$H = h\sin.^2\delta\,.\cos.2\theta + h'\sin.^2\delta'\cos.2\theta'.$$

But, in consequence of the retarding causes just spoken of, the pole of each spheroid will not be directly under the luminary, as it would be in the case of equilibrium, but will follow the luminary round the earth, and come to the meridian a certain time after the transit of the luminary. Hence, calling $\lambda$ and $\lambda'$ the angles which determine these times, the elevation at any point in the compound spheroid is

$$\tfrac{1}{2}H = \tfrac{1}{2}h\left\{\sin.^2\delta\cos.2(\theta - \lambda) + h'\sin.^2\delta'\cos.2(\theta' - \lambda')\right\}.$$

And this spheroid revolving round the earth, produces a flux and reflux at each place twice in 24 hours.

110. The angles $\theta$ and $\theta'$ go through a whole circumference in about 24 hours; and by this change $H$ twice at-

tains its maximum and twice its minimum in that time. The value of $\theta' - \theta$ remains nearly constant for this period: and the value of $\theta'$ for the maximum and minimum (the hours of high and low water) will be hereafter determined. (Art. 113.)

The height of high water varies with the value of $\theta' - \theta$. The highest high water is when the poles of the two spheroids coincide. (Art. 108.)

$$\text{In this case } \theta - \lambda = \theta' - \lambda' \text{: hence } \theta' - \theta = \lambda' - \lambda;$$

$$\text{and } H = (h' \sin.^2 \delta' + h \sin.^2 \delta) \cos. 2(\theta' - \lambda'):$$

By the diurnal revolution $\theta' - \lambda'$ changes through a circumference, and the greatest and least values of $\frac{1}{2}H$ are when

$$\theta' - \lambda' = 0, \text{ and } \theta' - \lambda' = \pi.$$

The sum of these is the elevation of high above low water, or the whole tide; and it is $h' \sin.^2 \delta' + h \sin.^2 \delta$.

The lowest high water is when the poles of the two spheroids are distant by a quadrant. In this case

$$\theta - \lambda = \theta' - \lambda' + \frac{\pi}{2}, \quad \cos. 2(\theta - \lambda) = -\cos. 2(\theta' - \lambda').$$

$$\text{Hence } H = (h' \sin.^2 \delta' - h \sin.^2 \delta) \cos. 2(\theta' - \lambda').$$

The whole tide in this case is $h' \sin.^2 \delta' - h \sin.^2 \delta$.

If we suppose the luminaries to be in the equator, or neglect the effects of declination, the proportion of the greatest to the least height of high water is $h' + h$ to $h' - h$.

This is the proportion of spring tides to neap tides. Tides of each kind occur twice in one revolution of $\theta' - \theta$.

We suppose $\theta$, $\lambda$, $h$, $\delta$ to refer to the sun, $\theta'$, $\lambda'$, $h'$, $\delta'$ to the moon. Hence $h'$ is greater than $h$.

111. PROP. ***The Time, Height and Circumstances of the Tide at each place will be determined by local conditions.***

The whole terrestrial tide which would exist if the earth were entirely covered by water, would present nearly the same circumstances at all places. But the formation of such a tide requires a very large portion of the earth's surface (for example, 90° of longitude,) to be open sea. And the tides which exist in narrower seas, as the Atlantic and the northern seas, are derivative tides, which flow from the general tides in the southern seas. When the tide thus enters narrow seas, it is modified by the form of the shores and depth of the channel, and takes considerable and various intervals of time in reaching different points. The general tide would travel from east to west, but the Atlantic tide turns *northward*, and travels from south to north; so that it is high water at nearly the same moments of absolute time on the coast of Guinea and Brazil, of Spain and New York. As it advances farther north, the tide turns *eastward* through the English Channel, and is 7 hours in travelling from Brest to Dover. Another wave of this tide travels northward on both sides of Ireland, and so reaches the North Sea.

Also, two tides may reach the same place by different courses. Thus, the tide-wave which enters the English Channel at 4 o'clock, reaches the North Foreland at 12; but the tide-wave which branches off from this on the west coast of Ireland, being in a more open sea, travels faster, reaches the Orkneys at 9 o'clock; then turning *southward*, moves along the east coast of Scotland and England, so that it is at Peterhead in Aberdeenshire at 12, and in 12 hours more it reaches the North Foreland, where it is met by the tide from the south, of 12 hours later origin.

In like manner, the tide which takes two hours only to sweep the western coast of Ireland, occupies 6 or 7 hours to pass the east coast of the same island, through St George's Channel; and the two tides meet in the neighbourhood of the Isle of Anglesea.

The peculiar circumstances of each place are represented in the formula by the particular values of $\lambda$ and $\lambda'$.

Thus it appears by Mr Lubbock's examination of the tide observations made at the port of London, (Companion to the Almanac, 1830. p. 59.) that for the London Docks we must have $\lambda = 3^h\ 29^m$, $\lambda' = 1^h\ 29^m$: $\lambda'$ referring to the moon and $\lambda$ to the sun. And hence, the two tides will coincide when the difference of Right-Ascensions of the sun and moon is 2 hours: that is after $\frac{1}{12}$ of a month, or $2\frac{1}{2}$ days from syzygy. Hence, the greatest tide will be $2\frac{1}{2}$ days after new and full moon.

It appears by the observations at Brest, that the highest tide arrives at that port very nearly a day and a half after the syzygy to which it corresponds. At this time the difference of the time of transits of the sun and moon is $1^h\ 15^m$ nearly. The solar tide is $4^h\ 24^m$ after the sun's transit, and the lunar tide is $3^h\ 9^m$ after the moon's transit. These coincide when the distance of the sun and moon is $1^h\ 15^m$ in time, and we then have the greatest tide.

112. Prop. *In general the Mean Retardation of the Tide will be equal to the Mean Retardation of the Moon's Transit:*

*But the least Retardation in a semi-lunation, will be when the Tides are greatest, and* vice versâ.

The moon comes to the meridian nearly 48 minutes later every day. The compound tide is intermediate between the lunar and solar tide, but nearer to the former, and coincident with it both at quadrature and syzygy. (Art. 108.) Hence, in the course of a month, the mean times of the tide will be regulated by the mean motion of the moon, and the mean retardation of the tide will be the same as that of the moon.

Since the compound tide is earlier than the lunar tide when the moon is going from syzygy to quadrature, and later than the lunar tide when the moon is going from quadrature to syzygy (Art. 108;) the tides before syzygy will be later than the mean time, and those after syzygy sooner than the mean

time. Hence, the intervals of the tides at syzygy are less than the mean intervals, and at quadratures they are greater. The moon passes the meridian later and later every day, the mean retardation being about 48 minutes a day, and the lunar tide being always at the same interval after every transit, would be retarded 48 minutes every day: but in consequence of the composition of the solar tide with the lunar, the retardation is least at syzygies, when the tides are greatest, and greatest at syzygies when the tides are least. The retardation of the tides on two successive days at syzygies is 39 minutes, at quadratures it is 75 minutes. These numbers, which are given by observation at Brest, are the same which result from theory. (Syst. du Monde, p. 278, and hereafter Art. 114.)

113. PROP. *To compare the results of theory and observation with regard to the Time of High Water.*

We have the expression already given for the height of the water at any moment, (Art. 109,)

$$H = h' \sin.^2 \delta' \cos. 2(\theta' - \lambda') + h \sin.^2 \delta \cos. 2(\theta - \lambda), \ldots (1)$$

to find where the water is highest, differentiate with regard to $\theta'$, and suppose $\frac{d\theta}{d\theta'} = 1$; (the value being really

$$\frac{\text{solar day}}{\text{lunar day}} \text{ or } \frac{29}{30} \text{ nearly}); \text{ hence,}$$

$$\frac{dH}{d\theta} = -2h' \sin.^2 \delta' \sin. 2(\theta' - \lambda') - 2h \sin.^2 \delta \sin. 2(\theta - \lambda).$$

At the point of maximum this is 0; therefore, at the moment of high water,

$$\sin. 2(\theta' - \lambda') = -\frac{h \sin.^2 \delta}{h' \sin.^2 \delta'} \sin. 2(\theta - \lambda).$$

Let $\frac{h \sin.^2 \delta}{h' \sin.^2 \delta'} = c$,

and $\theta' - \theta - \lambda' + \lambda = \phi$, whence $\theta - \lambda = \theta' - \lambda' - \phi$,

$$\sin.2(\theta' - \lambda') = -c\sin.2(\theta' - \lambda' - \phi)$$
$$= -c\sin.2(\theta' - \lambda')\cos.2\phi + c\cos.2(\theta' - \lambda')\sin.2\phi,$$

$$\text{hence } \tan.2(\theta' - \lambda') = \frac{c\sin.2\phi}{1 + c\cos.2\phi} \ldots(2).$$

Suppose the values of $c$, $\lambda$, $\lambda'$ known; we may assume different values of $\theta - \theta'$, that is, different ages of the moon; and hence from the value of $\phi = \theta' - \theta - \lambda' + \lambda$ we have the values of $\theta' - \lambda'$, and $\theta'$, that is the hour angle of the moon corresponding to high water.

Thus at the port of London, let the moon pass the meridian at 1 o'clock, therefore $\theta - \theta' = 1^h$: also (Art. 111,)

$$\lambda = 3^h\ 29^m,\ \lambda' = 1^h\ 29^m, \text{ whence } \lambda - \lambda' = 2^h: \text{ and } \phi = 1^h.$$

And (as will be shewn)

$$c = \frac{1}{2{,}6167}; \text{ hence, } \tan.2(\theta' - \lambda') = \frac{\sin.30^0}{2{,}6167 + \cos.30^0},$$

$$\text{whence } 2(\theta' - \lambda') = 9^0\ 20' = 37^m \text{ nearly.}$$

$$\text{Hence, } \theta' - \lambda' = 18\tfrac{1}{2}^m, \text{ and } \theta' = 1^h\ 46\tfrac{1}{2}^m \text{ nearly.}$$

It appeared by observation that when the moon passed at $1^h$, the tide followed at the interval of $1^h\ 47^m$. (*Lubbock*, Companion to the Almanac, 1830. p. 58.)

The agreement of the observed with the calculated times, (taking the mean of the observed times for a considerable period,) is found to be equally exact in the other cases.

The following is the table calculated by means of the above values:

| Time of Moon's Southing. | Time that the Moon's Southing precedes High Water. | |
|---|---|---|
| Hours. | Hours. | Minutes. |
| 0 | 2 | 0 |
| 1 | 1 | 47 |
| 2 | 1 | 32 |
| 3 | 1 | 17 |
| 4 | 1 | 4 |
| 5 | 0 | 55 |
| 6 | 0 | 54 |
| 7 | 1 | 6 |
| 8 | 1 | 32 |
| 9 | 1 | 58 |
| 10 | 2 | 10 |
| 11 | 2 | 9 |

*Lubbock*, p. 64.

114. Prop. *To compare the Retardation of the Tides at the times of the Moon's syzygy and quadrature.*

We have at high water, $\sin.2(\theta' - \lambda') = -c \sin.2(\theta - \lambda)$,

$$\text{and } \theta' - \theta - \lambda' + \lambda = \phi, \quad \theta' - \lambda' = \theta - \lambda + \phi,$$

whence $\sin.2(\theta - \lambda + \phi) = -c \sin.2(\theta - \lambda)$,

$$\sin.2(\theta - \lambda)\cos.2\phi + \cos.2(\theta - \lambda)\sin.2\phi = -c\sin.2(\theta - \lambda)$$

$$\tan.2(\theta - \lambda) = -\frac{\sin.2\phi}{c + \cos.2\phi} \ldots(3),$$

when $\phi = 0$, $\theta = \lambda$: and if $a$ be the semidiurnal increase of $\theta' - \theta$, we shall have at the next tide $\phi = a$, nearly, and

$$\tan.2(\theta - \lambda) = -\frac{\sin.2a}{c + \cos.2a}:$$

whence, if $a$ be a small arc (it is about $6\frac{1}{2}^{\circ}$) we shall have

$$\theta - \lambda = -\frac{a}{c+1} \text{ nearly}; \quad \theta = \lambda - \frac{a}{c+1} = \lambda - \frac{h'}{h'+h}a;$$

for, neglecting the effects of declination, $c = \frac{h}{h'}$.

Hence $\frac{h'}{h'+h}a$ is the alteration in the distance of the place of high water from that of the sun when $\phi = 0$; and, converted into time, is the semidiurnal retardation of the tide upon the sun.

When $\phi = 0$ or at syzygy, the tides are highest, hence the retardation at that period is $\frac{h'}{h'+h}a$.

When $\phi = \frac{\pi}{2}$ or at quadrature, the tides are smallest, and here also by (3)

$$\tan.2(\theta - \lambda) = 0, \quad 2(\theta - \lambda) = \pi, \quad \theta - \lambda = \frac{\pi}{2}.$$

And at the next tide, $\phi = \frac{\pi}{2} + a$ nearly; hence

$$\tan.2(\theta - \lambda) = \frac{\sin.2a}{c - \cos.2a}, \quad 2(\theta - \lambda) = \pi + \frac{2a}{c-1}, \quad \theta - \lambda = \frac{\pi}{2} - \frac{a}{1-c},$$

the retardation is $\frac{a}{1-c}$ or $\frac{h'}{h'-h}a$.

Hence, the retardation at spring tides is to that at neap tides as $h' - h$ to $h' + h$.

Cor. When the moon is in syzygy, $\theta' - \theta = 0$, $\phi = \lambda - \lambda'$,

$$\tan.2(\theta - \lambda) = -\frac{\sin.2(\lambda - \lambda')}{c + \cos.2(\lambda - \lambda')},$$

the hour of high water when the moon is in syzygy is called the *establishment of the port*. If $\epsilon$ be this quantity,

$$\tan. 2(\epsilon - \lambda) = -\frac{\sin. 2(\lambda - \lambda')}{c + \cos. 2(\lambda - \lambda')}.$$

115. PROP. *The two Tides on the same Day will be unequal, except the Sun and Moon be in the Equator.*

The action of the sun and moon produces the tidal spheroid. And the protuberance of this spheroid may be considered as composed of two tidal waves, one following the moon and the other opposite to this. When the sun and the moon are not in the equator, one of these may be called the northern and the other the southern tidal wave. And all places by their diurnal revolution are carried through both these tidal waves, and thus have high water twice in 24 hours. Now, places in the northern hemisphere pass nearer to the pole of the northern tidal wave, and places in the southern hemisphere nearer to the pole of the southern: and hence the tides are alternately greater and less. The greater tide, when the moon has *N* declination, the latitude being *N*, will be the superior tide, or that at the third lunar hour: when the moon's declination becomes *S*, this will be the smaller tide, (and vice versâ for *S* latitude.) And the greatest difference of the superior and inferior tide will be at the solstices, because then the poles of the tidal spheroid are most oblique to the equator: and still more if the ascending node of the moon be in the beginning of Aries, by which this obliquity will be further increased.

Thus in these latitudes, in winter the morning tide is larger than the evening tide, and in summer the evening tide is larger than the morning tide. The difference is about 1 foot at Plymouth and 15 inches at Bristol, according to Newton.

Also at Brest according to Laplace (Syst. du Monde, p. 86,) in the syzygies which occur about the summer solstice, the tides of the morning of the first and second day after the syzygy are smaller than the evening tide by about 7 inches; and greater by about the same quantity in the syzygies at

the winter solstice. In like manner, in the quadratures at the autumnal equinox, (when the sun is in the equator but the moon out of it), the morning tides of the first and second days after the quadrature exceed those of the evening by about 5 inches, and fall short of them by about the same quantity in the quadratures of the vernal equinox.

116. Laplace has treated the problem of the tides in a manner different from that above explained. He considers the tides as undulations which are excited and maintained in the fluid by the forces of the sun and moon, and which have periods of the same duration as the periods of the intensities of these forces. The tide which occurs twice in 24 hours is the first of the resulting undulations, the period being half a day. The second kind of undulation is one of which the period is a day, and this, combined with the preceding kind, causes the inequality of the two tides on the same day. He observes that the magnitude of the diurnal tide at any place, and its influence on the semidiurnal tide, will depend entirely upon the local circumstances: the former tides may be insensible; or they may increase so as entirely to render the latter insensible. It appears by calculation that the difference of the two tides of the same day will disappear if the depth of the ocean be everywhere the same. Again, under peculiar circumstances the difference of the two semidiurnal tides may itself become the tide, as in the following case.*

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* Laplace considers that Newton's explanation of the different magnitude of the two tides on the same day is erroneous, and observes that we may learn from this error to mistrust the most plausible trains of reasoning when not verified by strict calculation. (Syst. du Monde, p. 273). I do not conceive the objection thus made to Newton's views to be well founded. That view which Laplace substitutes for them is in fact the same reasoning under a different form. The diurnal periodic character of the forces which act upon the sea, consists in this, that the form of equilibrium which they would produce at intervals of a day is the same, but at intervals of half a day different. And whether we consider these forces as producing diurnal undulations according to laws of hydrodynamics, or consider the fluid as tending to the states of equilibrium which are thus possible, the effects on the circumstances of high and low water will be the same. Laplace says, that if Newton's explanation were the right one, the two semi-

Suppose two equal tides to come by different routes to the same port, one arriving 6 hours before the other, and occurring 3 hours after the moon's southing. If the moon, at this southing, was in the equator, there will arrive, every six hours, equal tides, which will supply each other's low water, and cause the water to remain without visible tide for a whole day. Now let the moon have declination; then the tides will be alternately greater or less, as has been shewn: and hence there will arrive at the port alternately two greater and two less tides. The two greater tides will produce the highest water at the intermediate time; between one greater and the next smaller tide the water will be at the mean height, and between the two smaller tides the water will be lowest. Thus in 24 hours there will not be two tides but one tide; and the greatest height, if for $N$ lat. the moon have $N$ dec. will be at 6 hours or at 30 ($= 24 + 6$) hours after the moon's southing; and when the moon has $S$ dec. the least height will occur at that time.

Such phenomena occur at the port of Batsham in Tunquin, in $N$ lat. $20^{\circ}\ 50'$. The two tides appear to come by the two channels which run, one from the China seas between the Continent and the island of Luconia; the other from the Indian sea between the Continent and the island of Borneo. (Newton.). The phenomena are described by Mr Davenport in the Phil. Trans. Vol. XIV. p. 677.

117. PROP. *To find the Force of the Sun to disturb the form of the ocean.*

(Newton, Book III. Prop. XXXVI.)

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semidiurnal tides at Brest should be very unequal instead of very nearly equal. But the great disparity of these tides would occur only if the globe were entirely covered with water, whereas in fact the tides which we have here, are derivative tides from the general tide in the southern seas, modified by their transmission along the whole length of the Atlantic.

It appears in the text that the very instance in which Laplace shews that the diurnal tide may completely cover and conceal the semidiurnal (that at Batsham) is deduced in the most complete and satisfactory manner from Newton's mode of treating the matter; and Newton himself has so deduced it.

The force of the sun to disturb any particle of the ocean will be exactly the same as the force of the sun to disturb the motion of the moon, taking account of the different distance of the disturbed body from the earth's centre. If $q$ be the periodic time of a body revolving in a circle at the earth's surface by the force of gravity, $P$ being the period of the earth about the sun, and $g$ the force of gravity, we shall have for the disturbing force on any point, tending to the centre of the earth, $\frac{gq^2}{P^2}$ as in Art. 76; and if $\omega$ be the angle made, by the radius of the earth at the point, with the line joining the centres of the earth and sun, $\frac{3gq^2}{P^2}\cos.\omega$ is the disturbing force parallel to this latter line. At quadratures the former is the only disturbing force, and tends to the earth's centre: from quadrature to syzygy there is a force accelerating the motion of a particle which revolves in that direction: at syzygy there is a force $\frac{2gq^2}{P^2}$ acting from the centre. Hence the total difference of the force at a point under the sun and at a point $90^0$ from this, will be $\frac{3gq^2}{P^2}$; and this is the force which disturbs the form of the spheroid.

If $a$ be the radius of the earth, $r$ the distance of the sun from the earth, and $\frac{m}{r^2}$ the force of the sun on the earth, we have, since $F \propto \frac{R}{P^2}$, $\frac{m}{r^2} : g :: \frac{r}{P^2} : \frac{a}{q^2}$;

$$\text{and the sun's disturbing force} = \frac{3gq^2}{P^2} = \frac{3ma}{r^3}.$$

In like manner if $r'$ be the distance of the moon from the earth, and $\frac{m'}{r'^2}$ the moon's attraction, $\frac{3m'a}{r'^3}$ is the moon's disturbing force.

$$\text{But we have } F = \frac{V^2}{r} = \frac{(2\pi r)^2}{P^2 r} = \frac{4\pi^2 r}{P^2};$$

$$\text{hence } \frac{m}{r^2} = \frac{4\pi^2 r}{P^2}, \quad \frac{3ma}{r^3} = \frac{12\pi^2 a}{P^2}.$$

If $P$ be measured in seconds, $g$ is the velocity acquired in one second. If $a = 3985$ miles $= 3985 \times 5280$ feet,

$$P = 365\tfrac{1}{4} \times 24 \times 60 \times 60,$$

we have, for the solar disturbing force in feet,

$$\frac{12 \times (3{,}1416)^2 \times 3985 \times 5280}{(365\frac{1}{4} \times 24 \times 60 \times 60)^2} = {,}0000025 \text{ feet, nearly};$$

and the proportion of this to gravity, is 1 to 12868200, nearly.

Cor. 1. If we suppose the spheroid acted on by the solar disturbing force to assume the form of equilibrium, the protuberance at its pole will be determined by the formulæ of Art. 106, and we shall have

$$\frac{h}{a} = \frac{5}{4} \times \frac{\text{disturbing force}}{\text{gravity}} = \frac{15ma}{4gr^3} = \frac{15\pi^2 a}{Pg};$$

$h$ being the difference of the semi-axes of the spheroid. And if we suppose the fluid to *tend* to the form of equilibrium, the elevation of the protuberance will still approximate to the same quantity. Hence, it appears that the sun would produce an elevation of the water at the pole of the tidal spheroid, amounting to 2 feet nearly.

Cor. 2. Hence, $C$ being a constant quantity, we shall have $h = \frac{mC}{r^3}$; and in like manner $h' = \frac{m'C}{r'^3}$; and equation (1) of Art. 113. becomes

$$H = C\left\{\frac{m}{r^3}\sin.^2\delta\cos.2(\theta - \lambda) + \frac{m'}{r'^3}\sin.^2\delta'\cos.2(\theta' - \lambda')\right\}.$$

118. PROP. *To find the Force of the Moon to disturb the form of the ocean.*

(NEWTON, Book III. Prop. XXXVII.)

The force of the moon will produce effects on the ocean of the same kind as the effects of the force of the sun, and the proportion will be found from the proportion of the effects.

It has been seen (Art. 110,) that the heights of the spring tides and of the neap tides are in the proportion of $h' + h$ to $h' - h$. According to Newton, in the river Avon, three miles below Bristol, the whole ascent of the water at the vernal and autumnal syzygies is about 45 feet; at the quadratures it is 25 feet. Hence, $h' + h : h' - h :: 45 : 25 :: 9 : 5$ and $h' : h :: 7 : 2$.

At Plymouth the mean tide is 16 feet, and the excess of the height at syzygies above that at quadratures is 7 or 8 feet. Hence the greatest and least tides are about 20 and 12, or $h' + h : h' - h :: 5 : 3$, whence $h' : h :: 4 : 1$, which is not much different from the former proportion.

Newton adopts the former value, or $c = \frac{2}{7} = \frac{1}{3,5}$, which makes the effect of the moon $3\frac{1}{2}$ times as great as that of the sun.

It is also shewn, Art. 112, that the retardation of the tide at syzygies and quadratures, is in the proportion $h' - h$ to $h' + h$. Now, in the table of the times of high water at London, p. 196, it appears that the greatest retardation of the tide is when the moon passes the meridian at 8 hours; for then, while the moon is retarded 1 hour, the tide is retarded 1 hour 26 minutes. Also, the least retardation is when the moon passes at 8 hours; for then, while the moon is retarded 1 hour, the tide is retarded 1 hour minus 15 minutes, or 45 minutes. Hence, we have, by this observation,

$$h' + h : h' - h :: 96 : 45 \text{ and } h' : h :: 131 : 41,$$

$$\text{whence } c = \frac{1}{3,2} \text{ nearly.}$$

Neither of the above methods is very accurate. By observations of the tide at Brest, Laplace calculated the ratio of the forces of the sun and moon on the ocean, and found

$$c = \frac{1}{2,6167}.$$

Cor. 1. The greatest height of the solar tide was found to be 2 feet nearly; and if we suppose the lunar tide to be 3 times as great, the whole tide will be about 8 feet; which, modified by the causes already pointed out, is sufficient to produce the appearances which are observed.

Cor. 2. The force of the moon is 3 times that of the sun, and therefore $\frac{1}{4289400}$ of gravity. This is too small to be detected by observations on pendulums. It only becomes sensible in its effect on the ocean.

119. Prop. *To determine the Mass of the Moon.*

We have (Art. 117,)

$$h = \frac{Cm}{r^3}, \quad h' = \frac{Cm'}{r'^3}; \quad \text{whence } c = \frac{h}{h'} = \frac{m}{m'} \cdot \frac{r'^3}{r^3}.$$

Also, the Horizontal Parallax of the sun and moon, or the angle which the earth's radius subtends at the luminary, is inversely as the distance. Hence, if $\Pi$ and $\Pi'$ be the parallaxes,

$$c = \frac{m}{m'} \cdot \frac{\Pi^3}{\Pi'^3}.$$

It is found that $\Pi = 8''54$, $\Pi' = 57'34''$: and if we take Mr Lubbock's value of $c$,

$$\log.\frac{m}{m'} \cdot \frac{\Pi^3}{\Pi'^3} = \log. c = 9.52452; \text{ and we find}$$

$$\frac{m'}{m} = \frac{1}{21688000}.$$

The mass of the earth is $\frac{1}{354936}$ of that of the sun: hence, the mass of the moon is $\frac{1}{61,1}$ of that of the earth, according to this method of determining it.

But we may also determine the mass of the moon by knowing her periodic time: for by Art. 57, if $e$ be the mass of the earth,

$$\frac{e+m}{r'^3} : \frac{m+e+m'}{r^3} :: (\text{moon's per.})^2 : (\text{earth's per.})^2 :: 1 : (,0748013)^2.$$

Hence, omitting $e$ and $m'$ with respect to $m$,

$$\frac{m}{m'+e}\,\frac{r'^3}{r^3} = (\cdot 0748013)^2. \quad \text{And if}$$

$\log.\frac{m}{m'}\,\frac{r'^3}{r^3} = 9,52452$, we find $\frac{m'}{m'+e} = \frac{1}{59,8}$ and $\frac{m'}{e} = \frac{1}{58,8}$; which represents the mass of the moon as compared with the earth.

The moon's mass, determined from Laplace's observations on the tides at Brest, is $\frac{1}{74,94}$; from the effects of precession it appears to be $\frac{1}{75}$.

120. Prop. *The Height of the Tide is affected by changes of the Declination and Parallax of the Sun and Moon.*

The value of $H$, as already given, depends on $\sin.^2 \delta$ and $\sin.^2 \delta'$; and is less as each of these arcs $\delta$ and $\delta'$ deviates from a quadrant. This is the result of theory for the general tide; and it is found that the tides at particular places exhibit in some degree the effect of this condition.

The effects of the sun and moon depend upon their distances; and the smaller the distance the greater is the effect, and this in the proportion of the inverse cubes of the distance,

as has already been seen. This is the same as the direct proportion of their parallaxes, or of their apparent diameters.

Hence, in winter, when the sun is in perigee, his effect is greatest; and hence, the tides are then somewhat greater in syzygy, and less in quadrature, than they are in summer. In like manner the moon, at the part of her lunation when she is in perigee, causes greater tides than she does 15 days sooner or later when she is in apogee. Hence, the greatest tides of all, which arise from the syzygy of the sun and moon both in perigee, cannot occur at two successive syzygies.

121. Prop. *To find the Figure of the Moon.*

(Newton, Book III. Prop. xxxviii.)

If the body of the moon be fluid, like our ocean, the attraction of the earth will produce upon it effects similar to those which the moon produces upon the earth: that is, it will form it into a prolate spheroid, of which the axis will be turned to the earth. And the force by which the earth thus affects the moon, will be to the force by which the moon affects the ocean, as the earth's attraction to the moon's attraction, and as the moon's radius to the earth's radius jointly: that is, as the earth's mass × moon's diameter to the moon's mass × earth's diameter. Taking $\frac{1}{75}$ for the proportion of the moon to the earth, and $\frac{100}{365}$ for the proportion of the diameters, we have

$$\frac{\text{earth's force on moon}}{\text{moon's force on earth}} = \frac{75 \times 100}{365} = 20{,}5.$$

Now, the moon produces on the earth an elevation of about 6 feet: hence, the earth would produce in the moon an elevation of the surface of about 123 feet: and the diameter of the moon which is directed towards the earth will be greater than the diameters perpendicular to this by 246 feet. This, therefore, is the form of equilibrium of the moon, and that

which she must from the first have had a constant tendency to assume, till it was attained.

Cor. It is a consequence of this spheroidal form of the moon, that the same face is always turned to the earth; for if the moon were in any other position, there would be a tendency to return to this position, and she would return to it, pass it, and oscillate about it till it was attained. But the forces which produce this oscillation being small, the oscillations must be very slow; and hence, when the unequal angular motion of the moon about the earth, combined with her equable angular motion about her axis, causes her face to be turned somewhat away from its mean position, the force is not strong enough to draw the pole of the spheroid into the direction of the earth immediately. The face of the moon is constantly turned very nearly towards that focus of her elliptical orbit, in which the earth is not; for the angular velocity of a body revolving in an ellipse about a force in one focus, is very nearly uniform about the other focus.

Hence arises the Libration of the Moon in Longitude, (Newton, Book III. Prop. XVII.)

# CHAP. VI.

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## UNIVERSAL GRAVITATION.

122. In the preceding chapters we have assumed the existence of forces varying according to some mathematical function of the distances from the points to which they tend; and have deduced the properties of the motion of bodies acted on by such forces. In the last chapter but one we have supposed several bodies to act upon each other upon the same assumption, and have calculated their motions. We have also, in illustration of this investigation, assumed that such actions take place between the earth, the sun and the moon, and have thus calculated the perturbations of the moon's elliptical motion which would take place. Finally, in the last chapter we have assumed the separate parts of the globe of the earth, and of the moon, to exercise the same kind of action, and have determined, upon this supposition, the leading circumstances in the equilibrated form of the solid masses, and in the motions of their fluid covering.

Many of the calculated results agree in a very remarkable manner with the phenomena of the earth and heavens, and thus give a great degree of probability to the assumed laws of action. But the evidence of the truth of these laws acquires a far higher kind of probability, when the laws upon which the observed phenomena must depend, are examined according to the order of their generality: the mode of action of different parts of the system in particular being first considered, and the laws thus separately inferred being afterwards compared with each other. It is then found that all these laws are but particular cases of one general law; that not only all the different

bodies in different parts of the universe exert the same kind of force on each other, but that every particle of each body also exerts the same kind of force; and that all the particular modes of action, which were before viewed as distinct, are only manifestations of one general mode of action, which prevails universally in every point of the material universe. This truth, incomparably the most comprehensive, elevated, and fertile in consequences, of any discovery ever made in the knowledge of external nature, was gradually more and more clearly apprehended, and at last fully established by Newton, about the years 1682 and 3. We shall point out the manner in which the different phenomena of the solar system lead us through a series of steps to this general doctrine.

123. Prop. *The Primary Planets (Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus,) revolve about the Sun in virtue of Forces which tend to the Sun, and are inversely as the squares of their distances from the centre.*

That the planets and the earth, in the order in which they are above enumerated, revolve about the sun in orbits nearly circular, is a doctrine established by Copernicus and his successors in philosophy, and capable of being proved by incontrovertible reasonings.

Attempts have been made to explain the motions of the heavenly bodies by means of material connections between them and the centres about which they move: such pieces of machinery were sometimes supposed to be transparent and solid, and were then conceived as *crystalline spheres*; or they were asserted to be fluid, and to revolve like a whirl of water or air, and were then called *vortices*. All these attempts were found to be quite inadequate to account for the movements of the stars such as they are observed; and the existence of such a frame-work of the heavens being also utterly unsupported by any evidence of the senses, there is no longer a single tenable argument for it.

We are left therefore to the remaining supposition, that the heavenly bodies revolve in a space which is empty, or

nearly empty of inert matter; and that their movements are regulated, not by material agents with which they are in contact, but by the forces of bodies acting upon them at a distance. They must therefore be governed by the laws of motion established in the previous part of this work; and the propositions already demonstrated concerning the effects of forces will apply to them.

**124.** If a planet were acted upon by no force, it would go on uniformly in a straight line. Hence a planet moving in its curvilinear path, must be perpetually deflected from the tangent by some force tending towards the side on which the path is concave.

The planets move about the sun nearly in circles, and nearly uniformly; and therefore they describe about the sun areas which are either nearly or exactly proportional to the times. Therefore (Introd. Prop. 2.) the forces by which they are acted upon, deflected from the tangents of their paths, and compelled to describe their orbits, tend either exactly or very nearly to the sun.

Let it be supposed that the forces tend exactly to the sun, the small error, if there be any, not being taken into the account at this stage of the investigation. We have then a number of bodies revolving in circles about the same centre: and the variation of the force which acts upon these bodies at different distances from the centre may be ascertained, knowing their periodic times and distances from the sun, by means of Prop. 4. Cor. 7. of the Introduction.

It was discovered by Kepler, and is called his *third* Law, that the squares of the periodic times are as the cubes of the mean distances from the sun. From this it follows, Introd. Prop. 4. Cor. 6, that the forces are inversely as the squares of the distances from the centre.

In order to see how nearly this law of force agrees with the phenomena, we have given the following Table. The first column contains the period of the planet in terrestrial days,

the fourth column contains the planet's distance from the sun (the *mean* distance, or semiaxis of the orbit, the orbit being an ellipse) the distance of the earth being 10. These numbers are taken from *Laplace*, Système du Monde, p. 122.

Let $P$ be the period, and $R$ the mean distance; then if the force vary inversely as $R^2$, we shall have $P^2 \propto R^3$: or $P^2 = kR^3$. Hence, $2 \log. P = 3 \log. R + \log. k$. Whence it appears that on the supposition of the truth of the law, $2 \log. P - 3 \log. R$ ought to be constant.

The second column of the Table contains $\log. P$, the third $2 \log. P$, the fifth $\log. R$, the sixth $3 \log. R$; the seventh contains $2 \log. P - 3 \log. R$, obtained by taking the difference of the numbers in the fourth and seventh columns.

It appears that for each of the planets this difference is very nearly 2,125, the logarithm corresponding to the number 13,3363.

| | Period. days. | log. $P$ | 2 log. $P$ | Semi-axis. | log. $R$ | 3 log. $R$ | Difference. |
|---|---|---|---|---|---|---|---|
| Mercury .. | 87,97 | 1,9443346 | 3,8886692 | 3,871 | 0,5878232 | 1,7634696 | 2,1251996 |
| Venus..... | 224,70 | 2,3516031 | 4,7032062 | 7,233 | ,8593185 | 2,5779555 | 2,1252507 |
| Earth..... | 365,26 | 2,5626021 | 5,1252042 | 10,000 | 1,0000000 | 3,0000000 | 2,1252042 |
| Mars ..... | 686,98 | 2,8369441 | 5,6735882 | 15,237 | 1,1828995 | 3,5486985 | 2,1251897 |
| Jupiter ... | 4332,58 | 3,6367466 | 7,2734932 | 52,028 | 1,7162371 | 5,1487113 | 2,1247819 |
| Saturn.... | 10758,32 | 4,0317445 | 8,0634890 | 95,388 | 1,9724937 | 5,9384811 | 2,1250079 |
| Uranus ... | 30688,71 | 4,4869786 | 8,9731572 | 191,833 | 2,2829233 | 6,8487699 | 2,1243879 |
| | | | | | | sum ..... | 14,8750219 |
| | | | | | number = 13,3363 | mean .... | 2,1250031 |
| | | | | | 13,3429 | greatest.. | 2,1252507 |
| | | | | | 13,3161 | least..... | 2,1243873 |

The greatest deviation of one ratio from another only amounts to $\frac{1}{500}$ of the whole: and this deviation may properly be referred to the slight inaccuracies in the observation of the elements of some of the orbits, to the motion of the planet about the common centre of gravity of itself and the sun, instead of about the centre of the sun; and to the mutual perturbations of the planets.

Hence the law of force above mentioned is the true law of force in this case, considering the orbits as *circles.*

The orbits however are not exactly circles, nor are the motions of the planets exactly uniform. It was discovered by Kepler and is called his *first Law,* that the orbit of Mars is an ellipse having the sun in the focus; another discovery of Kepler, his *second Law,* is that the motion of the planet at its perihelion is more rapid than the motion at the aphelion, in such a proportion that the area described about the sun at the two places are equal. This inference was afterwards extended to other parts of the orbit of this planet, and to the orbits of the other planets.

From the second Law it follows, Introd. Prop. 2, that the planets are retained in their orbits by forces tending to the sun. From the first Law it follows, by Prop. 11, that the force at different distances *in the same orbit* varies inversely as the square of the distance.

This law of variation of the force *in different orbits,* supposing them to be *ellipses* and not circles, also follows from the third Law of Kepler, by Prop. 15. For the mean distance given above is the same thing with the semiaxis major of the elliptical orbit, and the squares of these semiaxes in such orbits, when the force varies inversely as the square of the distance, are, by the Proposition just quoted, as the cubes of the periodic times.

If the law of force deviated at all from the inverse square of the distance, the angle between the apsides would be different from 180°, by Art. 53. Now it is found that the motion

of the perihelia of the planets is very slow, and therefore the deviation from the law of the inverse square is either none at all or very small.

Thus the secular sidereal motion of the perihelion of the earth's orbit is, 3646″ centigrade. This is in one year 36″ nearly. Hence the angle between two successive perihelia is 400,0036 instead of 400: or in Art. 53. $G : F :: 1{,}000009 : 1$. Hence it appears that the index of the force, in the case of different distances in the earth's orbit, does not differ by more than ,000000000018 from the number 2.

Hence it appears that both in the orbits of different planets and in different parts of the same orbit, bodies are actuated by a force tending to the sun and varying inversely as the square of the distances; and therefore in all parts of the solar system, we may conceive such a force to be diffused and to act upon the *primary* planets. And we may consider the Proposition above enunciated as established by induction, from the particular instances. It is Prop. 2, Book III. of the Principia.

125. This law was thus established by induction from the planets more anciently known, before the discovery of Uranus. But this new discovered planet, as appears from the table, agrees with the law as exactly as the others. In like manner the small planets still more recently discovered, Ceres, Juno, Pallas and Vesta, the orbits of which lie between those of Mars and of Jupiter, also agree with the law. These planets, besides differing from the others in being much smaller, differ also in having their orbits much more excentric and more inclined to the ecliptic than the larger ones; but the ratio of $P^2$ to $R^3$ deviates from the mean, little more or not at all more, than the others, as appears by the following table:

The excentricity of the earth's orbit is, 0168; whence the ratio of the least to the greatest distance from the sun is 29 to 30 nearly. The excentricity of

| | Venus, | the Earth, | Uranus, | Jupiter, | Saturn, | Mars, | Mercury, |
|---|---|---|---|---|---|---|---|
| is, | ,0068 | ,0168 | ,0467 | ,0482 | ,0561 | ,0933 | ,2055 |

But we find for the excentricities of

| Ceres, | Vesta, | Pallas, | Juno, |
|---|---|---|---|
| ,0785 | ,0891 | ,2416 | ,2543 |

respectively; and hence in the case of Juno the ratio of the least to the greatest distance is 3 to 5 nearly, or 18 to 30; and the mean distance of the earth being 10, the least and greatest distances are 20,6 and 34,7 respectively.

| | *P* | log. *P* | 2 log. *P* | *R* | log. *R* | 3 log. *R* | Difference. |
|---|---|---|---|---|---|---|---|
| Ceres .. | 1631,37 | 3,2256633 | 6,4513266 | 27,672 | 1,4420092 | 4,3260276 | 2,1252990 |
| Pallas.. | 1635,62 | 3,2267597 | 6,4535194 | 27,719 | 1,4427776 | 4,3283328 | 2,1251866 |
| Juno... | 1594,02 | 3,2024937 | 6,4049874 | 26,703 | 1,4265601 | 4,2796803 | 2,1253071 |
| Vesta .. | 1326,93 | 3,1228482 | 6,2456964 | 23,632 | 1,3735005 | 4,1205015 | 2,1251949 |

**126.** Prop. ***The Satellites of Jupiter and Saturn revolve about their respective Primaries in virtue of Forces which tend to the Primary, and are in each case inversely as the distances from the centre of the Primary.***

(Newton, Book III. Prop. i.)

The Satellites revolve round their primaries in orbits which are nearly circles, the centre of the primary being the centre. And the verification of the law of force here asserted must be conducted in exactly the same manner as was done for the primary planets revolving about the sun, in the last Article. The following Tables exhibit the calculation.

The deviation from the mean in the case of the third satellite of Saturn is so great that we are led to suppose that its distance from Saturn has been erroneously assigned. The exact determination of the elongations of the satellites of Jupiter and Saturn from their primaries by observation, is an object which has not, as yet, engaged much of the attention of modern astronomers.

**Jupiter's Satellites—Jupiter's equatoreal semi-diameter = 1.**

| | Period. days. | log. $P$ | 2 log. $P$ | Mean dist. | log. $R$ | 3 log. $R$ | Difference. |
|---|---|---|---|---|---|---|---|
| I. | 1,7691 | ,2477524 | ,4955048 | 6,0485 | ,7816477 | 2,3449421 | $\bar{2}$,1505627 |
| II. | 3,5512 | ,5503629 | 1,1007258 | 9,6235 | ,9833331 | 2,9499993 | $\bar{2}$,1507265 |
| III. | 7,1546 | ,8545854 | 1,7091708 | 15,3502 | 1,1861141 | 3,5583423 | $\bar{2}$,1508385 |
| IV. | 16,6888 | 1,2224253 | 2,4448506 | 26,9983 | 1,4313348 | 4,2940044 | $\bar{2}$,1508462 |

**Saturn's Satellites—Saturn's equatoreal semi-diameter = 1.**

| | Period. days. | log. $P$ | 2 log. $P$ | Mean dist. | log. $R$ | 3 log. $R$. | Difference. |
|---|---|---|---|---|---|---|---|
| I. | 0,9427 | $\bar{1}$,9743735 | $\bar{1}$,9487470 | 3,351 | ,5251744 | 1,5755232 | $\bar{2}$,3732238 |
| II. | 1,3702 | ,1367840 | ,2735680 | 4,300 | ,6334685 | 1,9004055 | $\bar{2}$,3731625 |
| III. | 1,8878 | ,2759560 | ,5519120 | 5,284 | ,7296288 | 2,1688884 | $\bar{2}$,3830236 |
| IV. | 2,7395 | ,4376713 | ,8753426 | 6,819 | ,8337207 | 2,5011621 | $\bar{2}$,3741805 |
| V. | 4,5175 | ,6548982 | 1,3097964 | 9,524 | ,9788194 | 2,9364582 | $\bar{2}$,3733382 |
| VI. | 15,9453 | 1,2026327 | 2,4052654 | 22,081 | 1,3440187 | 4,0320561 | $\bar{2}$,3732093 |
| VII. | 79,3296 | 1,8994352 | 3,7998704 | 64,359 | 1,8086093 | 5,4258279 | $\bar{2}$,3740525 |

The near approximation of the resulting differences to equality in each case proves the truth of the law.

Uranus has six satellites at present discovered, but the difficulty of observing them has prevented astronomers hitherto from obtaining data sufficient for the verification of Kepler's third law in this instance. Their greatest elongations from the planet have been observed, and the duration of their revolutions has been calculated by *assuming* the truth of the law. In the case of the second and fourth satellite however, which are the best known, these calculations are confirmed by observation.

127. PROP. *The Satellites of Jupiter and Saturn are attracted by the Sun with Accelerating Forces which are the same, at the same distances, as those by which their Primaries are attracted by the Sun.*

If a body $P$ revolve in a circle round a body $T$, and both be equally attracted by a distant body $S$, it has appeared, (Art. 64.) that $P$'s motion about $T$ will be alternately accelerated and retarded equally in opposite portions of its orbits, but that $T$ will still be the centre of the orbit. But if, while $T$ is attracted by $S$, $P$ be not attracted, or be less or more attracted than $T$, $P$ will no longer be equally accelerated and retarded in opposite portions of the orbit, but will be more or less affected in one portion of its orbit which is towards $S$ than in the opposite one which is from $S$; and the opposite portions will no longer be equally distant from $T$, or $T$ will no longer be in the centre of the orbit. And if the inequality of the action of $S$ on $T$ and on $P$ reach a certain limit, $P$ will no longer revolve about $T$ at all, but will be drawn towards $S$ if the attraction be greater on $P$, or will recede from $S$ if the action be less, and will describe a path which does not respect $T$ as a centre of revolution.

It appears by observation that the satellites of Jupiter and Saturn revolve in orbits which have the primaries for their centres exactly or very nearly. And it does not appear that

the motions of the satellites are affected by any inequalities which are of different amount on the side next the sun, and on the opposite side. Therefore the satellites are not acted on by any accelerating force towards the sun, which is, at equal distances, perceptibly different from the accelerating force by which the primaries tend towards the sun.

128. PROP. *The Moon is attracted by the Sun with an Accelerating Force which is the same, at the same distances, as that which the Sun exerts upon the Earth.*

This appears by the same reasoning as that which is employed in the last proposition; but the truth of the proposition can in this case be shewn more in detail than in that, because we can observe the inequalities of the moon's motion with great exactness. The Variation, and especially the motion of the Moon's Nodes, shew the action of the sun upon her. And it is found that these and other inequalities are the same, both as to their laws and their quantities, as those which were calculated in Sections 2 and 3 of Chap. IV., upon the supposition that the sun's force to disturb the relative motion of the moon round the earth depends entirely upon the different magnitude and direction of the distances of the moon and the earth from the sun. Thus the inequality called the *Variation* was collected from observation by Ptolemy, and was found to vary as the sine of twice the moon's distance from the sun in longitude, and consequently to vanish at syzygy and quadrature; and this inequality is found to depend upon the tangential disturbing force, and that force is thus shewn to vanish at quadrature; whence it appears that at quadrature, when the moon and earth are equally distant from the sun, the attraction of the sun upon them is equal. In like manner the *Evection* agrees with the inequality arising from the different rate of variation of the radial force when the perigee is in quadrature and in syzygy; and similarly the other inequalities of the moon's motion agree with the supposition of the sun exerting equal accelerating forces on the earth and moon, except in so far as these forces are modified by the different position and distances of the two bodies.

Therefore the earth and the moon are, at equal distances, equally attracted by the sun.

129. PROP. *The Force by which the Moon is retained in her orbit tends to the Earth, and varies inversely as the square of the distance from the Earth's centre.*

(NEWTON, Book III. Prop. III.)

The moon's apparent motion is found by observation to vary as the square of her apparent diameter: that is, the angle described about the earth in a given time varies as the square of the distance. Hence the area described in a given time is constant, and the force by which the moon is retained in her orbit, tends to the earth by Prop. 2. Introd.

The moon's perigee is found by observation to be nearly fixed, and therefore by Art. 53., the force varies nearly as the inverse square of the distance. The motion of the perigee in each revolution of the moon is only about $3^0$, and it appears from this, by Art. 53., that the index of the power according to which the force varies inversely, is $2\frac{4}{243}$. The difference from the index 2 arises from the action of the sun (see Art. 77.) and may here be neglected.

130. PROP. *The Force by which the Moon is retained in her orbit is the same with the Force of Gravity which acts upon terrestrial bodies.*

(NEWTON, Book III. Prop. IV.)

Let $P$ be the moon's periodic time, $R$ the mean radius of her orbit, $a$ the semidiameter of the earth.

Let the moon revolve in a circle round the earth at rest, at a mean distance $Q$ in the period $P$; and let $E$ be the mass of the earth and $M$ of the moon; $R$ being the mean distance when they revolve about the common centre of gravity, we have $Q = R\sqrt[3]{\dfrac{E}{E+M}}$. (Art. 57.) But if the force of gravity vary inversely as the square of the distance at the earth's surface, we have (Introd. Prop. IV. Cor. 10.)

Force by which $P$ is retained at distance $Q = \frac{4\pi^2 Q}{P^2}$;

and hence

Force at earth's surface

$$= \frac{4\pi^2 Q}{P^2} \cdot \frac{Q^2}{a^2} = \frac{4\pi^2 Q^3}{P^2 a^2} = \frac{4\pi^2 R^3}{P^2 a^2} \cdot \frac{E}{E+M}.$$

It appears from observation that the moon's mean parallax has for its value, independent of periodical changes, 56′ 55″. (Mac. Cal. Tom. I. p. 120.) which gives for the corresponding distance of the moon $R$, the value $a \times 60{,}4$. Also it has appeared, (p. 205.) that the value of $E$ is $M \times 58{,}8$. Moreover $P = 27{,}322$ days $= 27{,}322 \times 24 \times 60 \times 60$ seconds; and

$$a = 3985 \times 5280 \text{ feet}.$$

Hence the force of gravity at the earth's surface as deduced from the moon's motion will be

$$\frac{4\pi^2 (60{,}4)^3 58{,}8 \times 3985 \times 5280}{(27{,}322 \times 24 \times 60 \times 60)^2 \times 59{,}8} = 32{,}297 \text{ feet nearly.}$$

But the sensible gravity is diminished by the centrifugal force arising from the earth's rotation. At the equator this diminution is $\frac{1}{259}$ of the whole, (see p. 184.). Hence the sensible gravity will be 32,185 feet. And this is very nearly the force of terrestrial gravity. Hence the moon is retained in her orbit by the force of terrestrial gravity, diminished in the ratio of the inverse square of the distance.

This is often expressed by saying that the moon *gravitates* to the earth.

Cor. We may infer by analogy that the satellites of Jupiter, Saturn and Uranus, *gravitate* to their respective primaries. (Newton, Book III. Prop. v.)

131. Prop. *The Force by which bodies are attracted to the Earth arises from forces tending to each part of the mass, and varying inversely as the square of the distance.*

If the earth be a sphere homogeneous or concentrically heterogeneous, consisting of particles each of which exerts a force varying inversely as the square of the distance, the whole effect will tend to the centre of the sphere, and will vary inversely as the square of the distance by Art. 91. Hence the force which has been shewn to tend to the earth's centre, may, so far as can be collected from the law, either be a force tending to the central point, or the result of forces tending to each particle.

On either supposition the earth would, in consequence of the rotatory motion and the resulting centrifugal force, become an oblate spheroid. But on the former supposition the force of gravity would tend to the centre of the spheroid; on the latter it would be everywhere perpendicular to the surface. On the former supposition the length of the degrees of latitude would be greatest at the equator, and would diminish in going towards the poles: on the latter, the degrees of latitude would be least at the equator, and would diminish in proceeding towards the poles.

It has been ascertained by various measurements made in different countries, that the degrees of latitude are less in places near the equator than they are in more polar regions.

By Lambton's measures in India, the arc of the meridian from lat. $8^0\ 9'\ 38'',4$ to lat. $10^0\ 59'\ 48'',9$ is 1029100,5 feet.

By Svanberg's measures in Sweden the arc

from lat. $65^0\ 31'\ 32'',2$ to lat. $67^0\ 8'\ 49'',8$ is 593277,5 feet.

In the former case the arc $= 2^0\ 50'\ 10'',5 = 2^0,83625$,

and the mean length of a degree $= \dfrac{1029100,5}{2,83625} = 362838,4$ feet,

the middle of the arc being in lat. $9^0\ 34'\ 44''$.

In the latter, the arc $= 1^0\ 37'\ 17'',6 = 1^0,62155$,

and the mean length of a degree $= \dfrac{593277{,}5}{1{,}62155} = 3658706{,}7$ feet,

the middle of the arc being in lat. $66^{\circ}\ 20'\ 10''$.

Hence it appears that the form of the earth agrees with the supposition of a force tending to each particle, and not with the supposition of a force tending to the central point.

Again, it is found that masses of matter do exert a force of attraction which may under certain circumstances, be compared with the attraction of the earth.

Thus by astronomical observations made on the mountain Shehallien in Scotland, in which a plumb line was used, it was found that the directions of the plumb line on the north and south side of the mountain made a greater angle than they would have done in consequence of the rotundity of the earth, if the mountain had not been there. In each position the plumb line was drawn towards the mountain.

Also Mr Cavendish found that when a rod with a leaden ball at each end was suspended by a very fine thread, and other balls brought near those leaden balls, so that their attraction might tend to turn the rod horizontally, the rod was turned through a measurable angle, the attraction of the balls thus exerting a force sufficient to twist the thread.

Thus it appears that stone and metals do exert a force of attraction, and a globe like the earth, composed of such materials, would by this force cause bodies to tend to it. And the results of such a force would in all respects resemble those of the force which the earth exerts on the moon and terrestrial bodies. Such a force tending to the parts of the earth, is shewn to exist, and explains the phenomena; it is therefore the cause of those phenomena.

Cor. The force of attraction of the earth may, by the experiments just spoken of (those of Maskelyne and Cavendish) be compared with the force of attraction of a mountain, supposed to be a known mass of stone, or with the force of attraction of a given mass of lead.

The force of attraction of the earth being known as compared with that of a given mass, if we suppose the force of attraction to be as the mass or quantity of matter of the earth, we can find the whole quantity of matter of the earth.

Knowing the quantity of matter and the magnitude of the earth, we can find its density, as compared with the density of any known substance.

The density of the earth, and therefore its quantity of matter, was found to be nearly of the same value by the two sets of experiments above mentioned. By the calculations of Dr Hutton from Maskelyne's experiments the density was 5,3: by calculations from the experiments of Cavendish it was 5.

132. Prop. *Terrestrial Bodies are attracted by the Earth and by each other proportionally to their Quantities of Matter.*

(Newton, Book III. Prop. vi.)

The accelerating force is as the attraction or pressure directly, and as the quantity of matter inversely, by the third Law of motion. Hence, the attractions of the earth on different bodies are as the quantities of matter, if the accelerating force of gravity on all bodies be equal.

All bodies fall towards the earth with equal velocities, making allowance for the different effects of the resistance of the air.

The experiment can be made most accurately by means of pendulums; and Newton performed it in a manner which he describes as follows:

"I took two boxes of wood, round and equal. I filled the one with wood, and suspended an equal weight of gold (or other material) in the centre of oscillation of the other, as nearly as I could. These boxes, suspended by equal threads, eleven feet long, formed two pendulums, which, in regard of weight, figure, and the resistance of the air, were perfectly equal. Being put in oscillation near each other, they

swung to and fro in exact accordance for a very long period. Therefore the quantity of matter in the gold was to the quantity of matter in the wood, as the moving force on the gold, to the moving force on the wood; that is, as the weight of the one to the weight of the other. And so in the rest. I tried this in the case of gold, silver, lead, glass, sand, salt, wood, water, wheat. A difference in the quantity of matter which amounted to a thousandth of the whole, would be clearly detected in this manner."

Again, from the experiments of Maskelyne and Cavendish, mentioned in Art. 131, it appeared that the mass of the earth, obtained by comparison of its attraction with that of known bodies on each other, and by *supposing* their attraction proportional to their quantity of matter, was nearly the same when deduced from different bodies; from the mountain and from the leaden balls. Hence, the supposition that bodies of this kind attract each other with forces which are proportional to their quantities of matter, when compared with the earth and with each other, is true.

133. Prop. *The parts of the Earth are attracted by the Sun and the Moon.*

It has been shewn that a motion of the sea, agreeing in most of its circumstances with the phenomena of the tides, would result from the joint action of the sun and moon upon the waters of the ocean. We may therefore attribute those phenomena to that action.

It has been shewn also that a motion of the terrestrial sphere which would produce phenomena resembling the precession of the equinoxes, and of nutation, would result from the action of the sun and moon upon the protuberant meniscus by which the terrestrial spheroid exceeds the inscribed sphere. Therefore we may infer that these phenomena are the result of such actions.

134. Prop. *The Planets attract each other and the Sun.*

The planets produce inequalities in each other's motions, which follow from the supposition of their mutual attraction

in the same manner as the lunar inequalities follow from the supposition of the mutual attraction of the earth, moon, and sun.

Thus the perihelia of all their orbits advance, except that of Venus, which is retrograde. The inclinations of their orbits to the ecliptic also change; and their nodes upon the ecliptic are retrograde.

The amount of these perturbations is calculated by the Planetary Theory: and this theory, proceeding on the supposition that the disturbing planet attracts both the Sun and the planet disturbed, give, as results, certain inequalities in the planetary motions which agree in their law and quantity with observation.

135. Prop. *If in a system of any number of bodies* A, B, C, D, &c. *each body* A *attract the others* B, C, D, &c. *with accelerating forces which vary inversely as the square of the distance, the attractive powers of any two bodies* A, B *are as their quantities of matter.*

(Princ. Book I. Prop. lxix.)

By supposition, the accelerating forces of $A$ on $B$, $C$, $D$, &c. are equal, when $B$, $C$, $D$, &c. are at equal distances from $A$. In like manner, the accelerating forces of $B$ on $A$, $C$, $D$, &c. are equal at equal distances. Now the attractive power of $A$ is to that of $B$, as the accelerating force which $A$ exerts on $C$ is to that which $B$ exerts on $C'$ at the same distance. But the attraction or pressure which $A$ exerts on $B$, and that which $B$ exerts on $A$ are necessarily equal, being action and re-action. Hence, the accelerating force of $B$ on $A$ and of $A$ on $B$, are inversely as $A$ and $B$, or directly as $B$ and $A$. Also, the accelerating force of $A$ on $C$ is, by supposition, equal to that of $A$ on $B$; and the accelerating force of $B$ on $C$ is equal to the accelerating force of $B$ on $A$. Hence, it follows that the accelerating force of $A$ on $C$ is to that of $B$ on $C$ as $A$ to $B$; and therefore the attractive power of $A$ is that of $B$ as $A$ to $B$.

Cor. 1. Hence, if in the system of the universe it be found that the accelerating force exerted by each body on the others, is inversely as the square of the distance, and does not differ for different bodies attracted, the attractive power of each body is as its quantity of matter.

Cor. 2. The quantity of matter of each attracting body may hence be determined, by determining its attracting power from its effects on other bodies. And if the quantity of matter of any attracting body, thus determined, be found to be the same whatever be the attracted body which we consider, this identity is a proof that the attraction which takes place between the attracting and attracted body, is proportional to their quantities of matter alone, and does not vary with any peculiar properties of the different bodies.

If the mass of the attracting body were found to be of a different value, by calculations deduced from different attracted bodies, the inference to be drawn would be, that the attraction which takes place between the different bodies does depend in some measure upon some peculiar property of the matter of each, and not upon the quantity of matter alone.

Thus, it has been supposed by some astronomers, that the mass of Jupiter, as deduced from the perturbations of Saturn, is $\frac{1}{1070}$ of the mass of the Sun; but that the mass of the same planet, as deduced from the perturbations of Juno and Pallas, is about $\frac{1}{1054}$ of that of the Sun. If this difference should be confirmed by accurate observations and calculations, it would follow that the attractive power of the Sun and of different planets is not proportional to their quantities of matter alone. But the above masses cannot as yet be considered as determined with sufficient certainty and exactness to authorize any such conclusion.

136 Prop. *All portions of matter exert a Force of Attraction which is as the Quantity of Matter, and varies inversely as the square of the distance.*

This is the Law of UNIVERSAL GRAVITATION.

It has appeared that the Sun attracts the planets, (Art. 123,) that the planets attract the satellites, (Arts. 126, 129,) that the Sun also attracts the satellites, (Arts. 127, 128,) that the planets attract each other, (Art. 134.) It has appeared also, that the Sun and Moon attract the different parts of the Earth, (Art. 133,) and that the portions of terrestrial matter attract each other, (Art. 132.) A mutual attraction is thus found to prevail among all the portions of matter, both on the earth and in all parts of the universe, so far as we have the means, by observation or experiment, of tracing its effects. We infer, therefore, that it obtains universally.

The attraction which prevails between the larger portions of matter, agrees exactly with that which would result from the compound action of all the component small portions. The Sun, planets and satellites are spheres, or nearly spherical; and hence, an attraction of the particles varying inversely as the square of the distance, would have for its result a force tending to the centre of each, and varying inversely as the square of the distance, which is such a force as we find to exist.

In the case of the Earth, it has appeared (Art. 130,) that the attraction of the whole mass is the result of the joint action of all the parts; and that it is as the quantity of matter, (Art. 132,) agreeing with the supposition that all equal portions of matter exert equal forces.

The Earth, and the other globes which, like it, revolve round the Sun, are, so far as we can discover, of the same nature as the Earth and governed by the same laws. Thus, the form of Jupiter is perceptibly oblate as that of the earth is. Hence, we infer that the attraction which they exert is of the same kind, and depends on the same causes, as that which the earth exerts; and hence, that in those instances also the effect of the whole is the sum of the effects of the particular parts.

We find that the separate bodies of the system (planets and satellites), attract each other with forces which are in-

versely as the squares of the distances. If we suppose several such bodies to coalesce and form a new sphere, the attraction of this sphere will be composed of the attraction of its particular parts, and its effects will be in all respects like those of a planet.

Hence, we infer that all the parts of matter are governed by a Law of Universal Gravitation, and that gravity and the motions which take place among the bodies of the universe, are the results of this Law, thus affecting the component parts of the sun, earth, planets and satellites.

Cor. 1. It may be objected, that if this be true, terrestrial bodies ought perceptibly to attract each other. But, as has already been observed, their attraction is to the force of gravity as the mass of the body is to the mass of the Earth. The attraction is therefore much too small to be sensible without some peculiar contrivance.

Cor. 2. The force of gravity is of a different nature from the magnetic force. For the magnetic attraction is not proportional to the matter attracted. And some bodies are attracted more than others; while some are not attracted at all. Also, the force of magnetism in the same body may become more or less intense, and is often much greater than gravity in proportion to the quantity of matter.

137. In the preceding Induction of the Law of Universal Gravitation, the inference has been made by shewing that, 1°, the Laws to which we are led by all the phenomena of terrestrial gravity, 2°, the Laws which regulate the primary, and 3°, the secondary motions of the solar system, 4°, the various and complex Laws which regulate the minute perturbations of the movements and forms of the bodies of the system, and 5°, the Laws of certain phenomena which take place among terrestrial bodies in general (their mutual attraction) are all included in this one single Law. And this Law in general is nothing more than an extension of each particular Law; and each of the separate Laws is only a particularization of the general Law, according to the subject to which it is applied.

The Law of Universal Gravitation is the simplest General Law which can include all these particular Laws. But it may be possible to assume some other more complex Law which shall also include such of these particular Laws as are certainly established. Thus supposing the three first to be proved beyond doubt, it is possible to assume an hypothesis which shall verify these, and not verify $4^0$ and $5^0$. These latter Laws, depending upon the mutual perturbations of the planets, and the mutual gravitation of terrestrial bodies, require for their confirmation the greatest accuracy of observation and experiment, and may perhaps be considered as still susceptible of receiving further support from facts.

Assume the following *Hypothetical Law*.

Let bodies be composed of different elements *a*, *b*, *c*, &c., of which *a* attracts *a* only, *b* attracts *b* only, &c.; *a* not attracting *b*, &c. Let the Sun contain *equal* quantities of each of these elements, and let all the parts of the system of each planet (its body and its satellites) be in regard to their elements, *similarly* composed.

Let there be two bodies $X$ and $Y$, of which the first contains of the elements *a*, *b*, *c*, &c., the quantities

$$X_a,\ X_b,\ X_c, \ldots$$

and the second, the quantities $Y_a$, $Y_b$, $Y_c$, ... The attraction of the two bodies will be

$$X_a Y_a + X_b Y_b + X_c Y_c + \ldots$$

Hence, the accelerating force on the first body will be

$$\frac{X_a Y_a + X_b Y_b + X_c Y_c + \ldots}{X_a + X_b + X_c + \ldots}.$$

Now if $S$ be the Sun, by supposition $S_a = S_b = S_c = \&c. \ldots (1)$.

Also if $P$ be a planet, $I$, $II$, &c., its satellites, by supposition

$$P_a : I_a : II_a : \ldots = P_b : I_b : II_b : \ldots = P_c : I_c : II_c : \ldots = \&c.\ (2).$$

And if $E$ be the Earth, $M$ the Moon, $B$, &c., terrestrial bodies, by supposition

$$E_a : B_a : M_a : \ldots = E_b : B_b : M_b : \ldots = E_c : B_c : M_c : \ldots = \&c. \ (3).$$

1°. Hence, for the accelerating force of $S$ on $P$, we have

$$\frac{S_a P_a + S_b P_b + S_c P_c + \ldots}{P_a + P_b + P_c + \ldots} = S_a, \text{ by } (1).$$

Hence, the force of $S$ on different planets at the same distance is equal, and the reasoning of Art. 128, continues true.

2°. Again, for the accelerating force of $P$ on $I$, $II$, &c. for example,

$$\text{force of } P \text{ on } I = \frac{P_a I_a + P_b I_b + P_c I_c + \ldots}{I_a + I_b + I_c + \ldots}.$$

Let $P_a : I_a = 1 : \lambda$, hence, $I_a = \lambda P_a$,

and $I_b = \lambda P_b$, $I_c = \lambda P_c$, &c., by (2).

$$\text{force of } P \text{ on } I = \frac{\lambda P_a^2 + \lambda P_b^2 + \lambda P_c^2 + \ldots}{\lambda P_a + \lambda P_b + \lambda P_c + \ldots} = \frac{P_a^2 + P_b^2 + P_c^2 + \ldots}{P_a + P_b + P_c + \ldots},$$

which being independent of $I$, $II$, &c., is the same for all the satellites, and the reasoning of Art. 126, continues true.

3°. By exactly the same reasoning as that in the last case, and by supposition (3). Art. 130, would still be verified, namely, that the accelerating force upon all terrestrial bodies is equal, and is equal to that upon the satellite: and the same would be true for any primary.

4°. Yet in this case the accelerating force of a planet $N$ on two others $P$, $Q$ and on the Sun $S$ would not be equal. Thus the accelerating attraction of $N$ on $S$ is

$$\frac{S_a N_a + S_b N_b + S_c N_c + \ldots}{S_a + S_b + S_c + \ldots} = \frac{S_a (N_a + N_b + N_c + \ldots)}{S_a + S_b + S_c + \ldots}.$$

The accelerating attraction of $N$ on $P$ is

$$\frac{N_a P_a + N_b P_b + N_c P_c + \ldots}{P_a + P_b + P_c + \ldots}.$$

The accelerating force of $N$ on $Q$ is

$$\frac{N_a Q_a + N_b Q_b + N_c Q_c + \ldots}{Q_a + Q_b + Q_c + \ldots};$$

and these expressions, taken generally, are different from each other.

5°. Also for the accelerating attraction of a terrestrial body $B$, on another $C$, we have

$$B_b C_b + B_c C_c + \&c.$$

And if $B$ and $C$ be similarly constituted so that

$$C_c : C_b :: B_c : B_b,$$

we have the attraction

$$= C_b \frac{B_b^2 + B_c^2 + \&c.}{B_b} = C \cdot \frac{B_b^2 + B_c^2 + \&c.}{B}.$$

In like manner for the attraction of $B$ on $D$, on the same supposition, we have

$$D \cdot \frac{B_b^2 + B_c^2 + \&c.}{B}.$$

And these are as $C$ to $D$, that is, the attractions are as the bodies attracted.

But if the bodies $C$ and $D$ be not constituted similarly to $B$, this is not necessarily true.

The supposition made above concerning the constitution of bodies, resolves itself into the Newtonian Doctrine, if we suppose one element only.

138. Prop. *To compare the Masses of the Sun, the Earth, and the Planets which have Satellites.*

(Book III. Prop. viii. and Corollaries.)

Let $P$ be the period of the revolving body in days (planet or satellite), $R$ the number which represents its mean distance (see Arts. 123, 126,) and $\rho$ the unit of length in which this is

measured. Then by Introd. Prop. 4, Cor. 10, $F = \frac{4\pi^2 R\rho}{P^2}$, where $F$ is the force at the orbit of the revolving body. And at the distance 1 from the centre of the central body, this is increased in the ratio $(R\rho)^2 : 1$. Hence at the distance 1, the force is $\frac{4\pi^2 (R\rho)^3}{P^2}$: and by Art. 136 this is as the mass.

The logarithm is

$$2 \log. 2\pi + 3 \log. \rho - (2 \log. P - 3 \log. R).$$

In the case of the Moon $P = 27,322$, $R = 60,4$, taking the Earth's radius $= 1$.

$$\text{Hence } 2 \log. P - 3 \log. R = \bar{3},5299143.$$

The Sun's parallax is about $8'',72$: hence the Earth's radius being 1, the Sun's distance is $\frac{1}{\sin. 8'',72} = 23686$ nearly, and its logarithm is 4,3745054. Hence, since in Art. 124 the Sun's distance is 10, $\rho = 2368,6$ and taking $2 \log. P - 3 \log. R$ from Art. 124, we have for the log. of the number corresponding to the Sun's mass,

$$2 \log. 2\pi + 3\,(3,3745054) - 2,1250030,$$

$$\text{or } 2 \log. 2\pi + 7,9985132.$$

And for the Earth, the log. is

$$= 2 \log. 2\pi - \bar{3},4299143.$$

Hence we have 5,5284275 the logarithm of the ratio of the masses, and the mass belonging to the Earth being 1, the Sun's is 337620.

We have in the preceding calculations supposed the Moon to revolve round the Earth at rest. But if $E$, $M$

be the Earth and Moon revolving about their common centre of gravity, $S$, $T$ any other two bodies similarly revolving, $p$, $P$ the periods respectively, $r$, $R$ the distances; we have, by Art. 57, Cor. 1,

$$\frac{S+T}{E+M} \cdot \frac{r^3}{R^3} = \frac{p^3}{P^2}.$$

Or if $T$ may be neglected in comparison of $S$,

$$\frac{S}{E+M} \cdot \frac{r^3}{R^3} = \frac{p^2}{P^2}.$$

Hence, it appears that the ratio of the mass of $S$ above determined, is that which it bears to $E+M$.

Now (p. 205.) $E = 58{,}8 \times M$; hence increasing the above number in the ratio $E : E+M$, or $58{,}8 : 59{,}8$, we have 336590 for the ratio of the Sun's mass to the Earth's.

Again, for Jupiter; his radius $= 11 = \rho$; of which log. $= 1{,}0413927$; and taking the Satellite IV in Art. 127, the log. of the mass is

$$2 \log. 2\pi + 3(1{,}0413927) - \bar{2}{,}1508462 = 2 \log. 2\pi + 4{,}9733319,$$

and for the mass of the Earth and Moon, the log. is

$$= 2 \log. 2\pi - \bar{3}{,}5299143;$$

hence the log. of the ratio is $2{,}5032462$; and the log. of the ratio to the Earth alone is $2{,}5045737$, and the ratio $319{,}5$.

The log. ratio of the mass of the Sun to that of Jupiter is $3{,}0251813$, and the ratio $1059{,}3 : 1$.

In a similar manner the mass of Saturn may be found.

Cor. 1. To find the density of the Sun, Jupiter, or Saturn, we have density $= \dfrac{\text{mass}}{\text{content}}$; or if $a$ be the radius of the body,

$$\text{density} = \frac{\text{mass}}{\frac{4}{3}\pi a^3}.$$

Hence, log. density = log. mass − log. $\frac{4\pi}{3}$ − 3 log. $a$.

And the log. mass being found as in the proposition, we have hence the ratio of the densities.

Thus for Jupiter

$$\text{log. density} = 2{,}5032462 - \text{log.}\,\frac{4\pi}{3} - 3\,\text{log.}\,11,$$

for the Earth

$$\text{log. density} = \quad 0 \quad - \text{log.}\,\frac{4\pi}{3} - 3\,\text{log.}\,1.$$

Hence the log. ratio is $\bar{1}{,}3790681$ and the ratio = ,2397, so that Jupiter's density is less than one quarter of the Earth's.

Cor. 2. To find the superficial gravity of the Sun, Jupiter, or Saturn: supposing $a$ the radius,

$$\text{force at surface} = \text{force at dist. } 1 \times \frac{1}{a^2}, \text{ and} \propto \frac{\text{mass}}{a^2}:$$

and hence the superficial gravity in different cases will be compared with that at the Earth's surface, by means of the preceding calculations.

$$\text{Thus, log. } \frac{\text{gravity at Jupiter's surface}}{\text{terrestrial gravity}} = 2{,}5032462 - 2\,\text{log.}\,11$$
$$= ,4204608,$$

and the number corresponding to this is 2,6331.

This is the proportion in which the weight of a body would be increased, if transferred from the Earth's surface to that of Jupiter.

139. Prop. *To determine the Mass of the Earth by a comparison of the rate of oscillation of a pendulum at the surface and at a point below the surface.*

Let $a$ be the radius of the Earth, (supposed spherical,) $b$ the depth of the lower pendulum station, (which is supposed

to be a small fraction of the earth's radius). Also let $n$ be the mean density of the spherical shell which is above the lower station, $m$ the mean density of the sphere which is below the lower station. And let $1 - c : 1$ be the ratio of the rates of the same pendulum at the surface and at the lower station.

The mass of the internal sphere will be $\frac{4\pi m}{3}(a-b)^3$, and of the superficial shell the mass will be $\frac{4\pi n}{3}\{a^3 - (a-b)^3\}$. Each of these will attract with a force which may be expressed by the mass, divided by the square of the distance from the centre. Therefore the attraction on a point at the surface is

$$\frac{4\pi m}{3}\frac{(a-b)^3}{a^2} + \frac{4\pi n}{3}\left\{a - \frac{(a-b)^3}{a^2}\right\}.$$

Also the attraction of the superficial shell on a point at the lower station is nothing, (Art. 90.) Therefore the attraction there is

$$\frac{4\pi m}{3}(a-b).$$

Hence the ratio of the attraction at the surface to that at the lower station is

$$= \frac{m(a-b)^3 + na^3 - n(a-b)^3}{ma^2(a-b)} = \left(1 - \frac{n}{m}\right)\left(1 - \frac{b}{a}\right)^2 + \frac{n}{m}\left(1 - \frac{b}{a}\right)^{-1}$$

$= 1 - \frac{2b}{a} + \frac{n}{m}\frac{2b}{a} + \frac{n}{m}\frac{b}{a}$, omitting powers of $\frac{b}{a}$, which is small;

$$= 1 - \frac{b}{a}\left(2 - \frac{3n}{m}\right).$$

But the ratio of these attractions, or of the force of gravity at the two stations, is the ratio of the squares of the times of oscillation of the same pendulum: that it is, $(1-c)^2 : 1$ or $1 - 2c : 1$, $c$ being small.

Hence $2c = \frac{b}{a}\left(2 - \frac{3n}{m}\right)$ and $\frac{m}{n} = \frac{1,5}{1 - \frac{ca}{b}}$.

Also the mean density of the whole sphere

$$= \frac{\text{mass}}{\text{content}} = \frac{m(a-b)^3 + n\{a^3 - (a-b)^3\}}{a^3}$$

$$= (m-n)\left(1 - \frac{b}{a}\right)^3 + n = m - (m-n)\left(\frac{3b}{a} + \&c.\right)$$

$$= m \text{ nearly.}$$

Suppose the depth of the lower station to be $\frac{1}{20000}$ of the Earth's radius ( = 1000 feet nearly :) then $\frac{a}{b} = 20000$.

If a seconds pendulum gain $p''$ per day at the lower station

$$1 - c : 1 :: 86400 + p : 86400.$$

Hence $c = \frac{p}{86400}$; and $\frac{ca}{b} = \frac{p \times 200}{864} = p \times ,23$.

If the pendulum gain $1''$ per day, we find thus the ratio of mean density to the density of the superficial stratum

$$\frac{m}{n} = \frac{1,5}{1 - ,23} = 1,9 \text{ nearly.}$$

If the pendulum gain $2''$ a day

$$\frac{m}{n} = \frac{1,5}{1 - ,46} = 2,8 \text{ nearly.}$$

If the pendulum gain $3''$ a day

$$\frac{m}{n} = \frac{1,5}{1 - ,69} = 4,8 \text{ nearly.}$$

If the pendulum gain $4''$ a day

$$\frac{m}{n} = \frac{1,5}{1 - ,92} = 18,8 \text{ nearly.}$$

If the pendulum gain 4,32 seconds a day, $\frac{m}{n}$ is infinite: that is the diminution of gravity is just so much as is due to the change of distance from the centre, and the superficial stratum produces no effect in comparison with the central mass.

If the Earth be supposed not to be perfectly spherical or regular, the formula will still be applicable; for the attraction of the parts of the superficial stratum in the neighbourhood of the pendulum stations will be much greater than that of all the irregular elevations and depressions in other parts, by which the figure differs from a sphere.

The rate of oscillation of the pendulum at the upper and lower station may be compared, by comparing the rate of each pendulum with that of a clock placed near it, and comparing the two clocks by means of chronometers carried from the one to the other.